IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for exercise 4.6.5 in Pitman

Question a) The probability $P(X_i \leq x) = x$ since X_i is uniformly distributed. The number N_x of X_i's less than or equal to x follows a binomial distribution $bin(n, x)$ since the X_i are independent. The event $\{X_{(k)} \leq x\}$ corresponds to $\{N_x \geq k\}$. We get

$$P(X_{(k)} \leq x) = P(N_x \geq k) = \sum_{i=k}^{n} \binom{n}{i} x^i (1-x)^{n-i}$$

Question b) From the boxed result at the bottom of page 327 we have that ($X_{(k)}$ has $beta(k, n-k+1)$ distribution. Substituting $r = k$ and $s = n-k+1$ we get

$$P(X_{(k)} \leq x) = \sum_{i=r}^{r+s-1} \binom{r+s-1}{i} x^i (1-x)^{s+r-i-1}$$

which is the stated result.

Question c) The $beta(r, s)$ density is

$$f(x) = \frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1} = \frac{1}{B(r,s)} x^{r-1} \sum_{i=0}^{s-1} \binom{s-1}{i} (-x)^i$$

Now

$$P(X_{(k)} \leq x) = \int_0^x f(x) \mathrm{d}x = \int_0^x \frac{1}{B(r,s)} u^{r-1} \sum_{i=0}^{s-1} \binom{s-1}{i} (-u)^i \mathrm{d}u$$

$$= \frac{1}{B(r,s)} \sum_{i=0}^{s-1} \int_0^x \binom{s-1}{i} (-u)^{r+i-1} \mathrm{d}u = \frac{x^r}{B(r,s)} \sum_{i=0}^{s-1} \binom{s-1}{i} \frac{(-x)^i}{r+i}$$

as was to be proved.