IMM - DTU

02405 Probability
2003-10-29
BFN/bfn

Solution for exercise 4.5.7 in Pitman

Question a) The exercise is closely related to exercise 4.4.9 page 310, as it is the inverse problem in a special case. We apply the standard change of variable method page 304

$$Y = \sqrt{T}, T = Y^2, \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\sqrt{t}}$$

$$f_Y(y) = 2\lambda \cdot y e^{-\lambda y^2}$$

a Weibull distribution. See e.g. exercise 4.3.4 page 301 and exercise 4.4.9 page 310.

Question b)

$$\int_0^{\infty} 2\lambda y^2 e^{-\lambda y^2} \mathrm{d}y = \int_{-\infty}^{\infty} \lambda y^2 e^{-\lambda y^2} \mathrm{d}y$$

We note the similarity with the variance of an unbiased (zero mean) normal variable.

$$\int_{-\infty}^{\infty} \lambda y^2 e^{-\lambda y^2} \mathrm{d}y = \lambda \int_{-\infty}^{\infty} y^2 \sqrt{\frac{2\pi}{2\pi}} \sqrt{\frac{\frac{1}{2\lambda}}{\frac{1}{2\lambda}}} e^{-\frac{1}{2}\frac{y^2}{\frac{1}{2\lambda}}} \mathrm{d}y = \lambda \sqrt{\frac{\pi}{\lambda}} \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi}} \frac{1}{\frac{1}{\sqrt{2\lambda}}} e^{-\frac{1}{2}\frac{y^2}{\frac{1}{2\lambda}}} \mathrm{d}y$$

the integral is the expected value of Z^2, where Z is *normal* $\left(0, \frac{1}{2\lambda}\right)$ distributed. Thus the value of the integral is $\frac{1}{2\lambda}$ Finally we get

$$E(Y) = \sqrt{\lambda\pi} E(Z^2) = \sqrt{\lambda\pi} Var(Z)$$

$$= \sqrt{\lambda\pi} \frac{1}{2\lambda} == \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} = 0.51 \qquad \text{with } \lambda = 3$$

Question c) We apply the inverse distribution function method suggested page 320-323. Thus

$$U = 1 - e^{-\lambda X} \Rightarrow X = -\frac{1}{\lambda} \ln(1 - U)$$

Now $1 - U$ and U are identically distributed such that we can generate an exponential X with $X = -\frac{1}{\lambda}\ln(U)$. To generate a Weibull $(\alpha = 2)$ distributed Y we take the square root of X, thus $Y = \sqrt{-\frac{1}{\lambda}\ln(1 - U)}$.