Solution for exercise 4.4.9 in Pitman

Question a) Using the one-to-one change of variable results page 304 we get $Y = g(T) = T^\alpha$, $T = Y^{\frac{1}{\alpha}}$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha t^{\alpha-1}$

$$f_Y(y) = \lambda\alpha t^{\alpha-1} e^{-\lambda t^\alpha} \frac{1}{\alpha t^{\alpha-1}} = \lambda e^{-\lambda y}$$

the exponential density.

Question b) Once again using the one-to-one change of variable results page 304 we get $T = g(U) = (-\frac{1}{\lambda}\ln(U))^{\frac{1}{\alpha}}, U = e^{-\lambda T^\alpha}$, $\left|\frac{\mathrm{d}t}{\mathrm{d}u}\right| = \frac{1}{\lambda\alpha}\frac{1}{u}(-\frac{1}{\lambda}\ln(U))^{\frac{1}{\alpha}-1}$

$$f_T(t) = 1\frac{1}{\frac{1}{\lambda\alpha}\frac{1}{u}(-\ln(u))^{\frac{1}{\alpha}-1}} = \lambda\alpha e^{-t^\alpha}(t^\alpha)^{1-\frac{1}{\alpha}} = \lambda\alpha t^{\alpha-1}e^{-t^\alpha}$$

a Weibull(λ,α) density.

Alternative solution using cumulative distribution - section 4.5

Question a)

$$P(T^\alpha \le t) = P(T \le t^{\frac{1}{\alpha}})$$

Since T has the Weibull distribution we find

$$P(T \le x) = F_{Wei}(x) = \int_0^x \lambda\alpha u^{\alpha-1}e^{-\lambda u^\alpha}\mathrm{d}u = \left[e^{-\lambda u^\alpha}\right]_{u=0}^{u=x} = 1 - e^{-\lambda x^\alpha}$$

Now inserting $x = t^{\frac{1}{\alpha}}$ we get

$$P(T^\alpha \le t) = 1 - e^{-\lambda\left(t^{\frac{1}{\alpha}}\right)^\alpha} = 1 - e^{-\lambda t}$$

which shows us that T^α has an exponential distribution.

Question b) We examine the random variable $Y = (-\lambda^{-1}\ln(U))^{\frac{1}{\alpha}}$.

$$P(Y \le y) = P((-\lambda^{-1}\ln(U))^{\frac{1}{\alpha}} \le y) = P((-\lambda^{-1}\ln(U)) \le y^\alpha)$$

$$= P((\ln(U)) \ge -\lambda y^\alpha) = P(U \ge e^{-\lambda y^\alpha}) = P(1 - U \le 1 - e^{-\lambda y^\alpha})$$

Now since U is uniformly distributed so is $1-U$ and we deduce

$$P(Y \le y) = P(U \le 1 - e^{-\lambda y^{\alpha}}) = 1 - e^{-\lambda y^{\alpha}}$$

where the last equality follows from page 487 (cumulative distribution function), which was to be shown.