IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for exercise 4.4.10 in Pitman

Question a) First we introduce $Y = g(Z) = |Z|$ and note that $g()$ is strictly increasing on $]0,\infty[$, and strictly decreasing on $]-\infty,0[$. We then apply the formula in the box on page 304 and the many to one result on the top of page 307. In our case we have

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}, \qquad y = g(z) = |z|, \qquad \left|\frac{\mathrm{d}y}{\mathrm{d}z}\right| = 1$$

Inserting in the formula

$$f_Y(y) = \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}y^2} \qquad 0 < y < \infty$$

Question b) We introduce $Y = g(Z) = Z^2$.

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}, \qquad y = g(z) = z^2, \qquad z = \sqrt{y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = 2z = 2\sqrt{y}$$

Inserting in the boxed formula page 304 and once again using the many to one extension.

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}}e^{-\frac{y}{2}} \qquad 0 < y < \infty$$

This is a special case of the χ^2 distribution, here with 1 degree of freedom. The general case is introduced page 365. The distribution is extremely important in statistics (and probability).

Question c) We introduce $Y = g(Z) = \frac{1}{Z}$. With $Z = \frac{1}{Y}$ and $\frac{\mathrm{d}g(z)}{\mathrm{d}z} = -\frac{1}{z^2}$ we get

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\frac{1}{\frac{1}{z^2}} = \frac{1}{y^2\sqrt{2\pi}}e^{-\frac{1}{2y^2}}, -\infty < y < 0; 0 < y < \infty$$

Question d) We introduce $Y = g(Z) = \frac{1}{Z^2}$. With $Z = \frac{1}{\sqrt{Y}}$ and $\frac{\mathrm{d}g(z)}{\mathrm{d}z} = -2\frac{1}{z^3}$ we get

$$f_Y(y) = 2\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\frac{1}{\frac{2}{z^3}} = \frac{1}{y\sqrt{y}\sqrt{2\pi}}e^{-\frac{1}{2y}}, 0 < y < \infty$$

the factor 2 stems from the many to one situation page 306/307.