IMM - DTU

02405 Probability
2003-10-30
BFN/bfn

Solution for exercise 4.2.10 in Pitman

Question a) We define $T_1 = \int(T)$ such that

$$P(T_1 = 0) = 1 - P(T > 1) = 1 - e^{-\lambda}$$

using the survival function for an exponential random variable. Correspondingly

$$P(K = k) = P(T > k) - P(T > k+1) = e^{-\lambda k} - e^{-\lambda(k+1)} = e^{-\lambda k}\left(1 - e^{-\lambda}\right) = \left(e^{-\lambda}\right)^k \left(1 - e^{-\lambda}\right)$$

a geometric distribution with parameter $p = 1 - e^{-\lambda}$.

Question b)

$$P(T_m = k) = P(T > \frac{k}{m}) - P(T > \frac{k+1}{m}) = e^{-\lambda \frac{k}{m}} - e^{-\lambda \frac{k+1}{m}} = \left(e^{-\frac{\lambda}{m}}\right)^k \left(1 - e^{-\frac{\lambda}{m}}\right)$$

$p_m = e^{-\frac{\lambda}{m}}$.

Question c) The mean of the geometric distribution of T_m is

$$E(T_m) = \frac{1 - p_m}{p_m}$$

The mean is measured in $\frac{1}{m}$ time units so we have to multiply with this fraction to get an approximate value for $E(T)$

$$E(T) \tilde{=} = \frac{1}{m} E(T_m) = \frac{1 - p_m}{p_m}$$

$$= \frac{1}{m} \frac{e^{-\frac{\lambda}{m}}}{1 - e^{-\frac{\lambda}{m}}} = \frac{1}{m} \frac{1 - \frac{\lambda}{m} + o\left(\frac{\lambda}{m}\right)}{1 - \left(1 - \frac{\lambda}{m} + o\left(\frac{\lambda}{m}\right)\right)} \rightarrow \frac{1}{\lambda} \text{ for } m \rightarrow infty$$