IMM - DTU

02405 Probability
2003-10-15
BFN/bfn

Solution for exercise 4.1.4 in Pitman

Question a) The integral of $f(x)$ over the range of X should be one (see e.g. page 263).

$$\int_0^1 x^2(1-x)^2 \mathrm{d}x = \int_0^1 x^2 \left(\sum_{i=0}^{2} \binom{2}{i} (-x)^i \right) \mathrm{d}x$$

using the binomial formula $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.

$$\int_0^1 x^2 \left(\sum_{i=0}^{2} \binom{2}{i} (-x)^i \right) \mathrm{d}x = \sum_{i=0}^{2} \binom{2}{i} \int_0^1 (-x)^{i+2} \mathrm{d}x = \sum_{i=0}^{2} \binom{2}{i} (-1)^i \left[\frac{x^{i+3}}{i+3} \right]_{x=0}^{x=1} = \frac{1}{30}$$

such that

$$f(x) = 30 \cdot x^2(1-x)^2 \qquad 0 < x < 1$$

This is an example of the Beta distribution page 327,328,478.

Question b) We derive the mean

$$\int_0^1 x f(x) \mathrm{d}x = \int_0^1 x 30 \cdot x^2 \left(\sum_{i=0}^{2} \binom{2}{i} (-x)^i \right) \mathrm{d}x = 30 \sum_{i=0}^{2} \binom{2}{i} (-1)^i \left[\frac{x^{i+4}}{i+4} \right]_{x=0}^{x=1} = \frac{1}{2}$$

which we could have stated directly due to the symmetry of $f(x)$ around $\frac{1}{2}$, or from page 478.

Question c) We apply the computational formula for variances as restated page 261.

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_0^1 x^2 30 \cdot x^2 \left(\sum_{i=0}^{2} \binom{2}{i} (-x)^i \right) \mathrm{d}x = 30 \sum_{i=0}^{2} \binom{2}{i} (-1)^i \left[\frac{x^{i+5}}{i+5} \right]_{x=0}^{x=1} = \frac{30}{105}$$

such that

$$Var(X) = \frac{30}{105} - \frac{1}{4} = \frac{1}{28}$$

which can be verified page 478.

$$SD(X_{3,3})^2 = \frac{3 \cdot 3}{(3+3)^2(3+3+1)} = \frac{1}{28}$$