IMM - DTU

02405 Probability
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BFN/bfn

Solution for exercise 3.4.17 in Pitman

We introduce N as the number of children and D as the number of boys in a family. The number of boys in a family of size n is $bin\left(n, \frac{1}{2}\right)$ distributed. By applying the rule of averaged conditional probabilities we get

$$P(D=k) = \sum_{n=k}^{\infty} P(D=k|N=n)P(N=n) = \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{1}{2}\right)^n \cdot p^n(1-p)$$

The terms in the sum are close to the terms of a negative binomial distribution distribution see summary page 482 or derivation page 213 Example 4. We first identify the parameter r to be k+1. The probability of a succes is $\left(1-\frac{p}{2}\right)$. Summing over all possible outcomes $t \geq r$) for an $NB\left(k+1, \left(1-\frac{p}{2}\right)\right)$ distribution (using the distribution in the standard form -page 215 or page 482) gives

$$\sum_{m=0}^{\infty} \binom{m+(k+1)-1}{(k+1)-1} \left(1-\frac{p}{2}\right)^{k+1} \left(\frac{p}{2}\right)^m = 1.$$

Now using an appropriate change of summation variable $(n = m + k)$

$$\sum_{n=k}^{\infty} \binom{n}{k} \left(1-\frac{p}{2}\right)^{k+1} \left(\frac{p}{2}\right)^{n-k} = 1$$

We now need to manipulate our expression for $P(D=k)$ to apply this result thus eliminating or evaluating the sum.

$$P(D=k) = \sum_{n=k}^{\infty} P(D=k|N=n)P(N=n) = \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{1}{2}\right)^n \cdot p^n(1-p)$$

$$= \frac{(1-p)\left(\frac{p}{2}\right)^k}{\left(1-\frac{p}{2}\right)^{k+1}} \sum_{n=k}^{\infty} \binom{n}{k} \left(1-\frac{p}{2}\right)^{k+1} \left(\frac{p}{2}\right)^{n-k} = \frac{2(1-p)p^k}{(2-p)^{k+1}}$$