IMM - DTU

02405 Probability
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EFF,BFN/eff

Problem 1

By the Rule of Average Conditional Probabilities (p. 41), we find

$$P(D) = P(D|C)P(C) + P(D|C^c)P(C^c) = \frac{2}{9}\frac{3}{4} + \frac{4}{7}\frac{1}{4} = \frac{13}{42},$$

which is the 3rd option.

Problem 2

Let T_i denote the lifetime of the i'th power source. The total time T, in which the sensor has a working power source, is $T = T_1 + T_2$. This sum is gamma distributed (p. 289). We find

$$P(T > 2) = \sum_{k=0}^{1} \exp(-\lambda t)\frac{(\lambda t)^k}{k!} = \exp\left(-\frac{2}{3}\cdot 2\right) + \exp\left(-\frac{2}{3}\cdot 2\right)\left(\frac{2}{3}\cdot 2\right) = \frac{7}{3}\exp\left(-\frac{4}{3}\right),$$

which is the 4th option.

Problem 3

The expectation of a random variable is the expectation of the conditional mean (p. 403):

$$E(X) = E(E(X|Y)) = E(Y^2\varphi) = E(Y^2)\varphi.$$

To find $E(Y^2)$, we use the computational formula for variance(p. 186)

$$E(Y^2) = Var(Y) + (E(Y))^2 = \frac{1}{\gamma^2} + \frac{1}{\gamma^2} = \frac{2}{\gamma^2},$$

which gives

$$E(X) = \frac{2}{\gamma^2}\varphi.$$

This is the 5th option.

Problem 4

We use the formula for the expectation of a function of X (p. 263)

$$E(\sqrt{X}) = \int_0^1 \sqrt{x} 4x^3 dx = \frac{8}{9},$$

which is the 3rd option.

Problem 5

We consider the options one-by-one.

Option 1 is clearly non-sensical. Many examples can be given of real systems that have a continously increasing error probability.

Option 2 is wrong. The hazard is dependant on x and therefore varies with time.

Option 3 is wrong. The hazard rate is increasing with time, thus older lightbulbs have a larger error probability.

Option 4 is correct. See the explanation for option 3 above.

Option 5 is wrong. We are able to conclude 4).

Problem 6

We denote the event that a fly possess the genemutation as G, and the event that the test gives a positive outcome as T^+. We can then write the information given as

$$P(G) = \frac{1}{4}, \quad P(T^+|G) = 1, \quad P(T^+|G^c) = \frac{1}{4}.$$

Where G^c is the complementary event of G. Using bayes rule (p. 49), we find

$$P(G|T^+) = \frac{P(T^+|G)P(G)}{P(T^+|G)P(G) + P(T^+|G^c)P(G^c)} = \frac{1 \cdot \frac{1}{4}}{1 \cdot \frac{1}{4} + \frac{1}{4}\frac{3}{4}} = \frac{4}{7},$$

which is the 2nd option.

Problem 7

We insert the transformation (p. 451):

$$\begin{aligned} &P\left(\frac{3}{2}X + \frac{1}{2}\sqrt{3}Z \geq 0, X \geq X + \sqrt{3} \cdot Z\right) \\ =&P\left(Z \geq -\sqrt{3}X, Z \leq 0\right) \end{aligned}$$

Using rotational symmetry (see e.g. example 2, p. 457), we see the angle of interest is $arctan(\sqrt{3}) = \frac{1}{3}\pi$. The probability is then

$$\frac{\frac{1}{3}\pi}{2\pi} = \frac{1}{6},$$

which is the 5th option.

Problem 8

Consider it a success that the team wins the national soccer league. The distribution of the number of tries before the first success T, is geometrically distributed. We find (using p. 482)

$$P(T > 10) = \left(1 - \frac{1}{20}\right)^{10} = \left(\frac{19}{20}\right)^{10},$$

which is the 2nd option.

Problem 9

Denote the completion time of the i'th runner as X_i. The winning time, X_{min}, is the minimum of $X_1, \ldots, X_6$. We find, (using p. 319)

$$P(X_{min} < 13,5) = F_{min}(13,5) = 1 - (1 - F_X(x))^6.$$

Where $F_{min}(x)$ is the distribution function of the minimum, and $F_X(x)$ is the distribution function of the individual X_i's. We find $F_X(x)$ by standardization (p. 190)

$$P(X_{min} < 13,5) = 1 - (1 - \Phi(-1))^6,$$

which is the 3rd option.

Problem 10

Think of the problem as a sequence of events (section 1.6), where a certain sequence must occur for the game to end in round 4. Let D_i denote the event that the number drawn in the i'th round is different from all previously drawn numbers. We are interested in the series of events D_1, D_2, D_3, D_4^c. Using the multiplication rule for n events (p. 56), and the fact that $P(D_{i+1}|D_2 \ldots D_i) = P(D_{i+1}|D_i)$ (see p. 63), we find

$$P(D_1, D_2, D_3, D_4^c) = P(D_1)P(D_2|D_1)P(D_3|D_2)P(D_4^c|D_3)$$

The conditional probability $P(D_{i+1}|D_i)$ is $\frac{25-i}{25}$ (see p. 63). This gives

$$P(D_1)P(D_2|D_1)P(D_3|D_2)P(D_4^c|D_3) = \frac{24}{25} \cdot \frac{23}{25} \cdot \frac{3}{25},$$

which is the 3rd option.

Problem 11

If the coordinates of a point in the plane are standardized normally distributed, the distance to origo R is rayleigh distributed (see. p. 359). We find

$$P(R > 2) = 1 - \left(1 - \exp\left(\frac{1}{2}2^2\right)\right) = \exp(-2),$$

which is the 5th option.

Problem 12

We use the normal approximation to the binomial distribution (p. 99). For simplicity, we calculate the probability of 400 failures, rather than 600 wins.

$$P(N \geq 400) = \Phi\left(\frac{400 + 1/2 - 1000 \cdot 0.52}{\sqrt{(1000 \cdot 0.48 \cdot 0.52)}}\right).$$

Which is the 1st option.

Problem 13

Let N_i denote the number of insects consumed on the i'th flight. The total number of insect consumed on all 160 flights is $N = N_1 + N_2 + \ldots + N_{160}$. According to the central limit theorem (p. 106), this sum is approximately normal. We find

$$P(N > 50) = 1 - \Phi\left(\frac{50 - 160 \cdot \frac{3}{10}}{\sqrt{160 \cdot \frac{2}{5}}}\right) = 1 - \Phi\left(\frac{1}{4}\right),$$

which is the 1st option.

Problem 14

This a change of variable problem (see p. 304). We first ensure the transformation is one-to-one:

$$Y = -X^4 \Leftrightarrow X = \pm\sqrt[4]{-y}$$

But since $X > 0$ (as it is exponential), we can disregard the negative solution. We can then follow the remaining steps from p. 304,

$$\frac{dy}{dx} = -4x^3.$$

The density is then

$$\begin{aligned} f_Y(y) &= \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} \\ &= \frac{\exp(-x)}{4x^3} \\ &= \frac{1}{4}x^{-3}\exp(-x) \\ &= \frac{1}{4}\left((-y)^{\frac{1}{4}}\right)^{-3}\exp\left(-\sqrt[4]{-y}\right) \\ &= \frac{1}{4}(-y)^{-\frac{3}{4}}\exp\left(-\sqrt[4]{-y}\right). \end{aligned}$$

Which is the 4th option.

Problem 15

We consider it a success, if a child is immune towards measles. The number of immune children X is then described with a binomial distribution. We find the probability of 10 or more immune children by summing $P(X = 10) + P(X = 11) + \ldots ... + P(X = 19)$. These probabilities are found using the formula on page 81. We then see the correct answer is 3.

Problem 16

We first find the covariance of V and W

$$\begin{aligned} &Var(V - 2W) = 12 \\ \Longleftrightarrow &Var(V) + Var(-2W) + 2Cov(V, -2W) = 12 \\ \Longleftrightarrow &Var(V) + 4Var(W) - 4Cov(V, W) = 12 \\ \Longleftrightarrow &4 + 4 - 4Cov(V, W) = 12 \\ \Longleftrightarrow &Cov(V, W) = -1. \end{aligned}$$

The correlation is then found from the formula on page 432:

$$Corr(V, W) = \frac{Cov(V, W)}{sd(V)sd(W)} = -\frac{1}{2},$$

which is answer 4.

Problem 17

Denote the weight by W and height by H. We need to find

$$P(W > 65|H = 157)$$

We standardize the variables:

$$P(Y > 0|X = -1.5)$$

We then insert the transformation (see p. 451)

$$\begin{aligned}
&P(0.4 \cdot X + \sqrt{1-0.4^2} \cdot Z > 0|X = -1.5) \\
=&P(-0.4 \cdot 1.5 + \frac{\sqrt{21}}{5} \cdot Z > 0) \\
=&P\left(Z > \frac{-\sqrt{21}}{7}\right) \\
=&1 - P\left(Z \leq \frac{-\sqrt{21}}{7}\right) \\
=&1 - \left(1 - \Phi\left(\frac{\sqrt{21}}{7}\right)\right) = 0.2563,
\end{aligned}$$

which is answer 2.

Problem 18

The distribution of the second largest is Beta(2,2) (see bottom of page 327). Alternatively, we may find the density using the formula on top of page 327:

$$f_{(2)}(x) = 3\binom{2}{1}x(1-x) = 6x(1-x).$$

We then find the probability with an integral

$$P\left(X > \frac{1}{4}\right) = \int_{\frac{1}{4}}^{1} 6x(1-x)dx = \frac{27}{32},$$

which is the 4th option.

Problem 19

Using Markov's inequality (p. 174) we find

$$P(X \geq 60) = \frac{3}{60} = \frac{1}{20},$$

which is the 4th option.

Problem 20

We use the division formula (p. 424) to find the conditional distribution

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Thus we need the marginal distribution of Y, $P(Y = y)$. We use the formula on page 145 for this:

$$\begin{aligned} P(Y = y) &= \sum_{x=y}^{\infty} P(X = x, Y = y) \\ &= \frac{(\lambda\,(1-p))^y}{y!\,\mathrm{e}^{\lambda\,p}} \left(\frac{p}{1-p}\right)^y \end{aligned}$$

We then find the conditional distribution:

$$P(X = x|Y = y) = \frac{\mathrm{e}^{-\lambda}\,(\lambda\,(1-p))^x\,\mathrm{e}^{\lambda\,p}}{(x-y)!\,(\lambda\,(1-p))^y},$$

which is the 2nd option.

Problem 21

Let F denote the banks' funding gap, and T denote the duration of the recession. Denote by D the sum $F + T$. We are interested in the probability $P(D < 36)$. A sum of normally distributed variables is again normally distributed. We find the mean D:

$$E(D) = E(F) + E(T) = 12 + 18 = 30.$$

And the variance:

$$Var(D) = Var(F) + Var(T) + 2 \cdot Sd(F)Sd(T)Corr(F, T) = 4 + 25 + 2 \cdot 2 \cdot 5 \cdot 0.6 = 41.$$

The probability of interest is then

$$P(D < 36) = 1 - \Phi\left(\frac{36-30}{\sqrt{41}}\right) = \Phi\left(\frac{6}{\sqrt{41}}\right),$$

which is option 5.

Problem 22

The number of arrivals N in a time interval t is Poisson(λt) distributed, see page 289. We find

$$\begin{aligned} P(N > 3) =& 1 - P(N \leq 3) = 1 - ((P(N = 0) + P(N = 1) + P(N = 2) + P(N = 3)) \\ =& 1 - \left[\exp(-2)\left(1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!}\right)\right] = 1 - \frac{19}{3}\exp(-2), \end{aligned}$$

which is the 1st option.

Problem 23

The total area of the figure D is

$$D = 2 \cdot \frac{1}{2} 4 \cdot 2 = 8.$$

The area where $X < -1$ consists of 2 triangles of length and height 1. Their total area A is

$$A = 2 \cdot \frac{1}{2} 1 \cdot 1 = 1.$$

Since the point is chosen at random, the probability $P(X < -1)$ is equivalent to proportion where $X < -1$ in relation the entire area:

$$P(X < -1) = \frac{1}{8},$$

which is the 3rd option.

Problem 24

The sum $Z = X + Y$ can be found from the convolution formula on page 372. We find the integrand as

$$f(x, z - x) = \exp(-x - (z - x)) = \exp(-z).$$

This corresponds to both option 1 and 2. We find the limits by inserting $y = z - x$ in the domain of (X, Y) as well:

$$\begin{aligned} & 0 < \frac{x}{2} \leq z - x \leq 2x \\ \Leftrightarrow & 0 < \frac{3x}{2} \leq z \leq 3x \end{aligned}$$

This implies that

$$z \leq 3x \Rightarrow x \geq \frac{z}{3}.$$

And that

$$z \geq \frac{3x}{2} \Rightarrow x \leq \frac{2z}{3}.$$

This corresponds to the 2nd option.

Problem 25

Make some area considerations in the plane. For option 1, we first find the area $\{-\infty < X < x_2, -\infty < Y < Y_2\}$, then we substract the area $\{-\infty < X < x_1, -\infty < Y < Y_1\}$. This leaves the areas $\{-\infty < X < x_1, y_1 < Y < Y_2\}$ and $\{x_1 < X < x_1, -\infty < Y < Y_1\}$, thus, this option will yield a number too big.

In option 2 and 3, the area is the same as in option 1, except that two additional areas are added. Thus, these also yield too large a probability.

The answer must then be either 4 or 5. The question is whether to use $<$ or $\leq$. The distribution function includes values including (x, y). The lines at $X = x_1$ and $Y = y_1$ are substracted, and should not be included in the expression. Therefore, answer 4 is correct.

Problem 26

Let T denote the lifetime of the component. We then find

$$\begin{aligned} E(T) = E(E(T|X)) &= E\left(\frac{1}{X}\right) \\ &= \int_1^3 \frac{1}{x} f_X(x) dx \\ &= \frac{1}{2} \int_1^3 \frac{1}{x} dx \\ &= \frac{1}{2} \log(3) \end{aligned}$$

Which is option 1.

Problem 27

We multiply the density evaluated at x by the width of the interval (see page 263). This is the 1st option.

Problem 28

We start by determining the probability that $P(X = 4, Y = 8)$, i.e. the probability that the maximum of the two rolls is 4, while the sum is 8. Clearly, this can only happen when $(4, 4)$ is rolled. The probability of this is $1/36$. We have thus eliminated option 2 and 4. Furthermore, 3 and 5 are wrong because the probability $P(X = 1, Y = 8)$, i.e. the probability that the sum 8 is and the maximum is 1 is clearly zero. Hence option 1 is correct.

Problem 29

We use the formula at the bottom of page 349

$$E(XY) = \int_0^1 \int_y^1 8x^2y^2 \, dxdy = \frac{4}{9},$$

which is the 5th option.

Problem 30

Since X and Y are independant, we can use the multiplication rule for exepectation (p. 177). Using the linearity of the mean as well (p. 175) We find

$$\begin{aligned} E(Z) =& E(2X + 3)E(Y - 1) \\ &= (2E(X) + 3)(E(Y) - 1) \\ &= (2 \cdot 2 + 3)(5 - 1) = 28, \end{aligned}$$

which is the 5th option.