IMM - DTU

02405 Probability
2022-12-13
EFF,BFN/eff

Problem 1

Letting A be the event that there is a problem with the oil, and B the event that the lamp is on, we see that

$$P(A|B) = \frac{1}{5},\ P(B) = \frac{1}{15}$$

Therefore by the multiplication rule we get

$$P(AB) = P(A|B)P(B) = \frac{1}{5}\frac{1}{15} = \frac{1}{75}$$

which is the 3rd option.

Problem 2

If we let X be the random variable describing the distance from the center of the funnel, we see that X is standard Rayleigh distributed, and therefore we are in the situation of evaluating $P(X \geq 3)$ which we may do by looking up the tables on page 477, and using the CDF of the Rayleigh distribution to get

$$P(X \geq 3) = 1 - P(X \leq 3) = 1 - (1 - e^{1/2 \cdot 3^2}) = e^{-9/2}$$

which is the first option.

Problem 3

Let X be the random variable describing the number of tagged deer caught the second time. We see that X follows a hypergeometric without replacement distribution, with $G = 10, B = N - 10$. Then using the table on 477 we find

$$P(X = 5) = \frac{\binom{10}{5}\binom{N-10}{8-5}}{\binom{N}{8}}$$

which is the 4th option.

Problem 4

We have $E(X) = 4$, and we see that X is gamma distributed. Then we use the formula on page 286 to arrive at

$$P(X \geq 4) = e^{-4} \sum_{k=0}^{3} \frac{4^k}{k!} = \frac{71}{3} e^{-4}$$

which is the second option.

Problem 5

We consider one hundrede independent identically distributed random variables with $E(X_i) = 1.5$, and $\sigma(X_i) = 5$, and let S denote their sum. Then we wish to find

$$P(S \leq 300)$$

In order to do this we use the central limit theorem on page 196, thereby finding

$$P(S \leq 300) \approx \Phi\left(\frac{300 - 1.5 \cdot 100}{5 \cdot \sqrt{100}}\right) - \Phi(-\infty) = \Phi(3)$$

Which is the 4th answer.

Problem 6

Let E, B denote respectively the events that economy class and business class are fully booked. We are given $P(E \cup B) = \frac{1}{3}, P(B) = 1/8$. Therefore we see (potentially drawing a Venn diagram) that $P(E \cap B^c) = \frac{1}{3} - \frac{1}{8} = \frac{5}{24}$. Formally we could proceed by noting that $P(B^c) = P(E \cap B^c) + P(E^c \cap B^c)$. Then $P(E \cap B^c) = P(B^c) - P(E^c \cap B^c) = (1 - P(B)) - (1 - P(E \cup B)) = P(E \cup B) - P(B) = \frac{1}{3} - \frac{1}{8} = \frac{5}{24}$. the second option.

Problem 7

Let U, Y be continuous random variables, and $U \sim Uniform(0,1), Y|_{U=u} \sim \exp(u)$. Then using the integral condition formula on page 417, we find

$$P(Y > y) = 1 - P(Y \leq y) = 1 - \int_0^1 e^{-ux} \mathrm{d}x = \frac{1}{x} - \frac{1}{x} e^{-x}$$

Which is the 2nd option on the answer sheet.

Problem 8

Let A_i be the event that the i'th call fails. We wish to determine the probability that the first 5 calls fail $P(A_5A_4A_3A_2A_1)$, for this we use the multiplication rule and that we assume the calls are independent to find

$$P(A_5A_4A_3A_2A_1) = \frac{9^5}{10^5}$$

which is the 3rd option.

Problem 9

The random variable X has density $f(x) = xe^{-x}$, we define the random variable $Y = 1-e^{-X}$. We determine the density of Y by applying the formula for the transformation of densities on page 304

$$f_Y(x) = f_X(x) \cdot \left(\left|\frac{\mathrm{d}1 - e^{-x}}{\mathrm{d}x}\right|\right)^{-1} = xe^{-2x}$$

Then applying the transformation to y, we find

$$f_Y(y) = \frac{\log(y+1)}{(y+1)^2}$$

which we recognize as the 4th option.

Problem 10

Let (X, Y) follow a bivariate normal distribution, with $\rho = -\frac{7}{8}$. We wish to determine $P(X+Y \leq 1)$, and to do so we rewrite $Y = -\frac{7}{8}X + \sqrt{1 - \frac{49}{64}}Z$, where X, Z are independent. Inserting this, we find

$$P\left(\frac{1}{8}X + \frac{\sqrt{15}}{8}Z \leq 1\right) \Rightarrow$$
$$P(X + \sqrt{15}Z \leq 8)$$

We see that $X + \sqrt{15}Z \sim \mathcal{N}(0, 16)$, by the formula on page 363, and denoting this random variable as w we have $P(w \leq 8) = \Phi(2)$, the first option.

Problem 11

We see that the probability of the lotto player winning in at least n out of a 100 weeks is described by a negative binomial distribution, with $n = 100, p = 1/5$. Inserting this in the formula for negative binomial distribution on page 477 we get $P(X \geq 10) = 1 - P(X < 10)$, which if we insert is the 3rd option.

Problem 12

Let X, Y be uniform(0,1) distributed random variables, consider $P(X + Y \leq 1/2)$. We see by drawing this sum, see also p. 381, that this may be given as

$$\int_0^{\frac{1}{2}} x = \frac{1}{8}$$

which is the 5th option.

Problem 13

Let X_i be random variables describing the level of water in each of the 12 months. Then we wish to find $P(max(X_i) \geq 8)$. We see that the CDF of each X_i is given as $1 - e^{-x^{1/3}}$. Using the result on page 316, we get

$$P(max(X_i) \geq 8) = 1 - (1 - e^{-2})^{12}$$

which is the 5th option.

Problem 14

Let X, Y be random variables with joint density $f(x, y) = 4xy, 0 < x < 1, 0 < y < 1$. Then we see that we may write it as $f(x, y) = f(x)f(y), f(x) = 2x, f(y) = 2y$. Defining $Z = Y/X$, we wish to determine its density. Then we may use the formula for the quotient of two independent random variables on page 373 to find

$$\int 4x^3 z \mathrm{d}x$$

Drawing the box $0 < x < 1, 0 < y < 1$, and looking at the line determined by $X = Y/Z$ we find the integration limits to be $1/z$ for $1 < z$ and 1 for $0 \leq z \leq 1$. This gives us

$$\int_0^1 4x^3 z \mathrm{d}x = z$$
$$\int_1^{1/z} 4x^3 z \mathrm{d}x = \frac{1}{z^3}$$

the first option.

Problem 15

Let X_1, X_2, X_3 be multinomially distributed random variables, with $p_1 = 1/2, p_2 = 1/3, p_3 = 1/6$ and 3 trials. We wish to find $P(X_1 + X_3 \leq 2)$. We note that this corresponds to finding $P(W \leq 2)$ where $W = X_1 + X_3$ is binomially distributed, with $n = 3, p = 2/3$. We see by applying the CDF of the Binomial distribution on page 477 that we have option 1.

Problem 16

Let X, Y be continuous random variables, where $Y|_{X=x} \sim \exp(1/x)$ and $X \sim$gamma(2,1)ie. the density of X is $f(x) = xe^{-x}$. Then we use the formula for the conditional expectation on page 423, where we note that $g(Y) = Y$, so $E(Y|X = x) = x$. Then we get

$$E(Y) = \int x(xe^{-x})\mathrm{d}x = 2$$

which is the 2nd option.

Problem 17

Let $X_1, .., X_5$ be random variables describing the time for the emergency hotline with density $f(x) = 2xe^{-x^2}$. We wish to determine the 3rd order statistic of these variables. In order to do this we apply the formula on page 326

$$5(2xe^{-x^2})\binom{4}{2}(1 - e^{-x^2})^2(1 - (1 - e^{-x^2}))^2 = 60xe^{-3x^2}(1 - e^{-x^2})^2$$

this we recognize as the 5th option.

Problem 18

The definition of independence p. 151 gives 3.

Problem 19

Let X be the random value describing the price of some option. We have $E(X) = 0, Var(X) = 49$. We find $P(X \geq 21)$ by applying Chebyshev's inequality, p. 191. To get

$$P(|X - E(X)| \geq 3\sigma(X)) \leq \frac{1}{3}$$

which we recognize as the 3rd option.

Problem 20

Let X, Y follow a standardized bivariate normal distribution with $\rho = -1/2$. We wish to determine $P(X \leq 0, Y \leq 0)$ and in order to do so we apply the method of page 457. We write

$$P\left(X \leq 0, \frac{1}{2}X + \sqrt{1-1/4}Z \leq 0\right) = P\left(X \leq 0, \frac{X}{\sqrt{3}} \leq Z\right)$$

Drawing this, we see that we get the angle $\pi/2 - \arctan\left(\frac{1}{\sqrt{3}}\right)$, and thus the probability $P(X \leq 0, Y \leq 0) = \frac{\pi/2 - \arctan\left(\frac{1}{\sqrt{3}}\right)}{2\pi} = 1/6$, the third option.

Problem 21

The joint density of the random variables X, Y is given as $f(x, y) = 3e^{-(x+y)}, 1/2x \leq y \leq 2x$ and we wish to determine $E(\sqrt{YX})$. To do this we apply the formula on page 423 to get

$$\int_0^\infty \int_{1/2x}^2 x\sqrt{xy}3e^{-(x+y)}\mathrm{d}y\mathrm{d}x$$

which we recognize as the 3rd option.

Problem 22

Let X the random variable describing the time of an oak trees death. We wish to find the probability of an oak tree dying in a short timeframe after 100 years, assuming that the density describing the lifetime is $f(t) = 3/(1+2t)^{5/2}$. In order to do this we calculate the hazard rate, page 297, which we find to be

$$\lambda(t) = \frac{f(t)}{1 - \int_0^t f(t)\mathrm{d}t} = \frac{3(2t+1)^{3/2}}{(2t+1)^{5/2}}$$

So $\lambda(100) = \frac{3}{201}$, and therefore the probability of dying within a short timeframe Δt is $\lambda(100)\Delta 3 = 1/201\Delta t$, the first option.

Problem 23

For two random variables X, Y we have that $E(X) = 2, V(X) = 4, E(Y) = 3, V(Y) = 9, E(XY) = 6$. We calculate the covariance by the formula on page 431 to get $Cov(X, Y) = E(XY) - E(X)E(Y) = 6 - 6 = 0$. Therefore by the formula for the correlation on page 433 they are uncorrelated, which was the 2nd option.

Problem 24

We have that $P(C_1) = P(C_2) = 1/4$, $P(C_3) = 1/2$, and $P(C_1 \cup C_2 \cup C_3) = 1$. Further $P(D|C_1) = P(D|C_2) = 1/5$, and $P(D|C_3) = 2/5$ and wish to calculate $P(C_1|D)$. In order to do this we use Bayes rule, page 49.

$$P(C_1|D) = \frac{P(C_1|D)P(C_1)}{P(C_1|D)P(C_1) + P(C_2|D)P(C_2) + P(C_3|D)P(C_3)} = \frac{1}{6}$$

which is the 2nd option.

Problem 25

We may assume that the prevalence of fungus follows a poisson distribution, since this may be seen as a poisson scatter, p. 228, where $\mu = 2 \cdot \frac{1}{4}$. Letting X be the random variable with this distribution, we see by the formulas on page 478 that $P(X = 0) = e^{-1/2} \approx 0.607$, the first option.

Problem 26

A discrete random variable N is uniformly distributed on $(1, \ldots, n)$, and the random variable $X|_{N=i} \sim$Bin(i,1/i) ie. it is binomially distributed with probability parameter $1/i$ and i trials. We wish to determine $P(X = n)$, and thus note that the probability of this happening is nil unless $i = n$. Therefore using the rule of average conditional probabilities on page 396 we get

$$P(X = n) = \frac{1}{n}\binom{n}{n}\frac{1}{n^n}\left(1 - \frac{1}{n}\right)^{n-n} = \frac{1}{n^{n+1}}$$

which we recognize as the 4th option.

Problem 27

We start by noting that our function is not a density on the negative reals, as $e^{-x} \to \infty$ for $x \to -\infty$. We recall that a density must be positive, and as such we require that the funcition we are given is positive on the interval we are looking at, and as such we must find the roots of $(x^2 - 4x + 3)$ if they are both negative then this may be extended to a density on the positive reals. However we see that $x^2 - 4x + 3 = 0 \Rightarrow x = 3 \vee x = 1$, which are both positive so the function is never a since the function must change signs at these two points. Thus we get option 5.

Problem 28

X and Y are two independent random variables following a geometric, respectively a Poisson distribution with parameters $p = 1/5, \mu = 2.$. We wish to find $E(XY) + E(X)E(Y)$, but since they are independent p. 177 gives $E(XY) = E(X)E(Y)$. Thus we have $2E(X)E(Y)$ and looking these up on page 477 gives us $2E(X)E(Y) = 20$, option nr. 4.

Problem 29

The distribution describing the number of failures before a success is precisely the negative binomial distribution, page 482, which is option nr. 5.

Problem 30

Let X, Y be random variables following a bivariate normal distribution with $\rho = 4/5$. We have $E(X) = 500, E(Y) = 200, V(X) = 2500, V(Y) = 1600$, and wish to determine $E(X|Y = 160)$. In order to do this we convert into standard units, where the standardized variables are denoted by $*$, and determine

$$Y = 160 \Rightarrow Y^* = \frac{Y - E(Y)}{\sigma(Y)} = \frac{-40}{40} = -1$$

Further we have by the formulas on the conditionals on page 451 that $E(X^*|Y^* = y^*) = \rho y^*$, so $E(X^*|Y^* = -1) = -\frac{4}{5}$. Converting back to X from X^* then gives

$$X = (X^*)50 + 500 = 50\frac{-4}{5} + 500 = 460$$

which is the 4th option.