DTU Compute

02405 Probability
30-10-16
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Problem 1

Let T_6 denote the arrival time of 6'th plane. This is gamma (Erlang) distributed (see the Poisson arrival process, p. 289). We can also interpret this probability as the probability of having between 0 and 5 arrivals (see p. 286). Thus

$$P(N \leq 5) = \sum_{i=0}^{5} P(N = i) = \sum_{i=0}^{5} e^{-\lambda t} \frac{(\lambda t)^i}{i!}.$$

We identify t as 2 (t being the number of time steps, here 30/15). Evaluating the sum gives answer 4.

Problem 2

We are asked to solve $F(x) = 0.75$. We get

$$1 - e^{-\left(\frac{\log\left(\frac{x}{\alpha}\right)}{\beta}\right)^2} = \frac{3}{4}.$$

Solving this equation for x yields option 1.

Problem 3

Let W, S and B_i denote the events drawing a white ball, drawing a black ball, and choosing box i, respectively. Summarizing the information from the assignment, we get

$$P(S|B_1) = \frac{1}{3}, P(W|B_1) = \frac{2}{3}, P(S|B_2) = \frac{1}{4}, P(W|B_2) = \frac{3}{4}, P(B_i) = \frac{1}{2}.$$

We are asked to find $P(B_1|W)$. By an application of Bayes rule (p. 49) we get

$$P(B_1|W) = \frac{P(W|B_1)P(B_1)}{P(W|B_1)P(B_1) + P(W|B_2)P(B_2)} = \frac{\frac{2}{3}\frac{1}{2}}{\frac{2}{3}\frac{1}{2} + \frac{3}{4}\frac{1}{2}} = \frac{8}{17}.$$

which is option 3.

Problem 4

X is exponentially distributed, meaning the density function is $f_X(x) = \lambda e^{-\lambda x}$. Using the definition of an expectation of a function of X (p. 263), we find

$$\begin{aligned} E(e^X) &= \int_0^\infty e^x \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{x(1-\lambda)} dx \\ &= \frac{\lambda}{1-\lambda} \left[e^{x(1-\lambda)} \right]_0^\infty \longrightarrow \infty \text{ for } (1-\lambda > 0). \end{aligned}$$

Thus, the mean is not defined for $1 - \lambda > 0 \Rightarrow \lambda < 1$. For $\lambda > 1$ we get

$$= \frac{\lambda}{1-\lambda}(0-1) = \frac{\lambda}{\lambda - 1}.$$

Which is option 3.

Problem 5

We find the hazard rate from the definition (p. 297)

$$\lambda(t) = \frac{f(t)}{G(t)} = \frac{\frac{d}{dx}F(x)}{1-F(x)} = \frac{xe^{-\frac{1}{2}x^2}}{e^{-\frac{1}{2}x^2}} = x$$

Which is option 1.

Problem 6

The inclusion-exclusion formula states

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Solving for $P(A \cap B)$, which is the probability that both A and B occurs, we get

$$P(A \cap B) = P(A) + P(B) - P(A \cup B).$$

Which is option 3

Problem 7

Let N denote the number of infected eggs. We are asked to find $P(N \leq 2)$. Since we have a limited number of experiments with a constant probability of success, N is binomially distributed (see p. 81). We find

$$P(N \leq 2) = \sum_{i=0}^{2} \binom{10}{i} \varphi^i (1-\varphi)^{10-i}.$$

Evaluating the sum gives option 2.

Problem 8

By an application of Bayes rule, we find

$$
\begin{aligned}
P(X \in [x; x+dx] | Y < \cos^2(\pi X)) &= \frac{P(Y < \cos^2(\pi X) | X = x) P(X \in [x; x+dx])}{P(Y < \cos^2(\pi X))} \\
&= \frac{\cos^2(\pi x) \cdot 1}{\int_0^1 \cos^2(\pi x) dx} \\
&= \frac{\cos^2(\pi x)}{\left[\frac{x}{2} + \frac{\sin(2\pi x)}{4\pi}\right]_0^1} \\
&= 2\cos^2(\pi x).
\end{aligned}
$$

Which is option 3.

Problem 9

This is sampling without replacement (see p. 125). For this probability we identify $G = 12, g = 0, B = 28, B = 5$. Inserting into the formula on p. 125 we get option 2.

Problem 10

Using the computational formula for variance (p. 186) and the linearity of the mean (p. 167), we find

$$
\begin{aligned}
Var(X+Y) &= E((X+Y)^2) - E(X+Y)^2 \\
&= E(X^2) + E(Y^2) + 2E(XY) - (E(X) + E(Y))^2
\end{aligned}
$$

Using the definition of expectation (p. 263) We find

$$
\begin{aligned}
E(X) &= \int_0^1 x \int_0^x 2 \, \mathrm{d}y \mathrm{d}x = \frac{2}{3} \\
E(Y) &= \int_0^1 y \int_y^1 2 \, \mathrm{d}x \mathrm{d}y = \frac{1}{3} \\
E(X^2) &= \int_0^1 x^2 \int_0^x 2 \, \mathrm{d}y \mathrm{d}x = \frac{1}{2} \\
E(Y^2) &= \int_0^1 y^2 \int_y^1 2 \, \mathrm{d}y \mathrm{d}x = \frac{1}{6} \\
E(XY) &= \int_0^1 \int_0^x 2xy \, \mathrm{d}y \mathrm{d}x = \frac{1}{4}.
\end{aligned}
$$

Inserting these quantities in the equation above gives option 2.

Problem 11

Draw the line $Y = X$ in the coordinate system. The area inside the half-circle and under the line is the desired area. The line $Y = X$ forms an angle of $\pi/4$ with the horizontal. The total angle around the half-circle is π. Thus, $P(Y < X) = \frac{\pi/4}{\pi} = \frac{1}{4}$, which is option 3.

Problem 12

The Poisson distribution is used to describe the number of counts, when there is no clearly defined upper limit on the number of events. Thus, option 2 is correct.

Problem 13

Let ACG_1 denote the event that the first triple is ACG, and TTA_2 that the second triple is TTA. Using the product rule, we find

$$P(ACG_1 \cap TTA_2) = P(TTA_2|ACG_1)P(ACG_1) = 0,06 \cdot 0.,14 = 0,0084.$$

This is option 2.

Problem 14

We have

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}, \quad \frac{dz}{dx} = 2x, \quad x = \pm\sqrt{z}.$$

Since there are two solutons to $Z = X^2$, we can not use one-to-one transformation of variables. Instead, we follow the approach from example 5, p. 307.
For $x \notin [-1;1]$, the numerator gives 0. For $x \in [-1;1]$, we get

$$f_Z(z) = \sum_{x=\pm\sqrt{z}} \frac{f_X(x)}{\left|\frac{dz}{dx}\right|} = \frac{\frac{1}{2}}{|2\sqrt{z}|} + \frac{\frac{1}{2}}{|-2\sqrt{z}|} = \frac{1}{\sqrt{2z}}.$$

This is option 5.

Problem 15

Let X denote the weight of a bag of soup. We first find the probability that one bag of soup contains less than 300g.

$$P(X < 300) = \Phi\left(\frac{300 - 305}{3}\right).$$

The number of bags with less than 300g, denoted N here, is binomially distributed with $n = 10$ and $p = \Phi\left(\frac{300-305}{3}\right)$. The probability that $N = 0$ is found, using the probability mass function for a binomial distribution (p. 81) we find

$$P(N = 0) = \binom{10}{0}\Phi\left(\frac{300-305}{3}\right)^0\left(1 - \Phi\left(\frac{300-305}{3}\right)\right)^{10-0} = \left(1 - \Phi\left(\frac{300-305}{3}\right)\right)^{10}.$$

which is option 1.

Problem 16

We have

$$f_X(x) = \lambda e^{-\lambda x}, \quad f_Y(y) = \begin{cases} \frac{1}{a} & \text{for } 0 < y < a \\ 0 & \text{otherwise.} \end{cases}$$

Using the convolution formula (p. 372), we find

$$f_Z(z) = \int_0^z f_X(x) f_Y(z-x)dx$$

The first function is straightforward to deal with, as it is defined for all $x \geq 0$. For the second we have the condition that $z - x < a$, leading to $x > z - a$. If $z < a$, this condition is irrelevant, since we alredy know that $x > 0$. But if $z > a$, we need to take it into account. Thus, for $z < a$, the integral becomes

$$f_Z(z) = \int_0^z \lambda e^{\lambda x}\frac{1}{a}dx = \frac{1}{a}\left(1 - e^{-\lambda z}\right).$$

For $z > a$, we get

$$\int_{z-a}^z \lambda e^{\lambda x}\frac{1}{a}dx = \frac{1}{a}(e^{\lambda a} - 1)e^{-\lambda z}.$$

Thus, the answer is option 4.

Problem 17

Let X denote the value of the die. We find (using p. 175)

$$E(X^2) = \sum_{x=1}^{6} x^2 P(X = x) = \frac{1}{6}\sum_{x=1}^{6} x^2 = \frac{91}{6}.$$

Which is option 5.

Problem 18

Let N denote the number of U-235 atoms. We use the Normal Approximation to the Binomial Distribution (p. 99)

$$P(N < 325) = \Phi\left(\frac{325.5 - 1200 \cdot 0.25}{\sqrt{1200 \cdot 0.25 \cdot 0.75}}\right) = \Phi(1.7).$$

This is option 4.

Problem 19

We find (using p. 193)

$$Var(X - 2Y) = Var(X) + 4Var(Y) = 30 + 4 \cdot 20 = 110.$$

which is option 1.

Problem 20

Let X denote the amount of biological material. By definition of probability densities (p. 263), we find

$$P(X \in [14; 14.01]) = f_X(14) \cdot \frac{1}{100} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(12-14)^2}{4}} \frac{1}{100} = \frac{1}{200\sqrt{2\pi}} e^{-\frac{1}{2}}.$$

This is option 3.

Problem 21

Define U and V as independent standard normal variables, i.e.

$$U = \frac{X-1}{2} \Rightarrow X = 2U + 1, \quad V = \frac{Y-1}{2} \Rightarrow Y = 2V + 3.$$

Inserting this into the equation for the circle gives

$$4U^2 + 4V^2 = 4 \Leftrightarrow U^2 + V^2 = 1$$

Thus, the problem is rewritten in the form of standard normal variables. The task was to find the probability of being outside the circle. The above equation specifizes a circle with center at $(0, 0)$ and radius 1. Thus, we need to find the probability that the point (X, Y) has distance greater than 1 to $(0, 0)$, i.e. that $R = \sqrt{X^2 + Y^2} > 1$. This distance is Rayleigh distributed (see p. 359). We find

$$P(R > 1) = e^{-\frac{1}{2}}.$$

This is option 5.

Problem 22

Using Bayes rule, we find

$$\begin{aligned}
P(X = x|X + Y = m) &= \frac{P(X + Y = m|X = x)}{P(X + Y = m)} \\
&= \frac{P(Y = m - x)P(X = x)}{P(X + Y = m)}
\end{aligned}$$

The sum of independent geometric variables with the same parameter is negative binomial (see p. 213). Thus, we find

$$\begin{aligned}
&= \frac{q(1-q)^{m-x-1}q(1-q)^{x-1}}{\binom{m-1}{1}q^2(1-q)^{m-2}} \\
&\frac{(1-q)^{m-2}}{(m-1)(1-q)^{m-2}} = \frac{1}{m-1}.
\end{aligned}$$

Which is option 5.

Problem 23

The number of tries N, before the first success, is geometrically distributed on $1, 2, \ldots$. We find

$$P(N = 3) = \left(1 - \frac{1}{20}\right)\frac{1}{20} = \frac{19^2}{20^3}$$

Which is option 3.

Problem 24

Let N denote the number of guests, and S denote the number of glasses used. We use that the expectation is the expectation of the conditional mean (p. 403), the rule of average conditonal expectations (p. 402) and the definition of the mean for a binomial distribution (p. 90) to find

$$\begin{aligned}
E(S) &= E(E(S|N)) \\
&= \sum_n P(N = n)E(S|N = n) \\
&= 0.2\frac{80^2}{40} + 0.5\frac{120^2}{40} + 0.3\frac{160^2}{40} \\
&= 101.
\end{aligned}$$

This is option 4.

Problem 25

Using Chebychev's inequality (p. 191) we find,

$$P(|X - E(X)| > 5 \cdot SD((X)) < \frac{1}{5^2} = \frac{1}{25}.$$

Which is option 2.

Problem 26

We use the formula on p. 346 to find

$$P\left(\frac{1}{4} \le X \le \frac{3}{4}, \frac{1}{4} \le Y \le \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{x}^{\frac{3}{4}} 6(y-x)\mathrm{d}y\mathrm{d}x.$$

Which is option 4-.

Problem 27

Let N denote the number of activations. We find

$$P(N \ge 8) = 1 - P(N < 8) = 1 - \sum_{i=0}^{7} \frac{e^{-4}4^i}{i!} = 0.0511$$

This is option 1.

Problem 28

Let X denote lung capacity and Y denote blood pressure. Using the transformation on p. 451, we find

$$\begin{aligned} P(X < 0, Y < 0) &= P\left(X < 0, \frac{-\sqrt{2}}{2}X + \sqrt{1 - \left(\frac{-\sqrt{2}}{2}\right)^2} Z\right) \\ &= P\left(X < 0, \frac{\sqrt{2}}{2}X > \frac{1}{\sqrt{2}}Z\right) \qquad = P(X < 0, Z < X) \end{aligned}$$

Using rotational considerations (see example 2, p. 457), we see the angle of interest is $\frac{\pi}{4}$. Thus, $P(X < 0, Z < X) = \frac{\frac{\pi}{4}}{2\pi} = \frac{1}{8}$, which is option 1.

Problem 29

Let X and Y be standardized versions of L and V, respectively. We find

$$\begin{aligned} P(V > 90|L = 1) &= P\left(X > \frac{90-85}{3}\middle| Y = \frac{1-1.2}{0.1}\right) \\ &= P\left(X > \frac{5}{3}\middle| Y = -2\right) \end{aligned}$$

Using the result for conditionals on p. 451, we get $(X|Y = -2) \sim N(-\frac{1}{2}, \frac{15}{16})$. Thus

$$= 1 - \Phi\left(\frac{\frac{5}{3}+\frac{1}{2}}{\frac{\sqrt{15}}{4}}\right)$$

Which is option 5.

Problem 30

Inserting the transformation (p. 451), we find

$$\begin{aligned} P(X + Y < 1) &= P\left(X + \frac{1}{2}X + \sqrt{1-\frac{1}{2^2}}Z < 1\right) \\ &= P\left(\frac{3}{2}X + \frac{\sqrt{3}}{2}Z < 1\right) \end{aligned}$$

The mean is $E\left(\frac{3}{2}X + \frac{\sqrt{3}}{2}Z < 1\right) = \frac{3}{2}E(X) + \frac{\sqrt{3}}{2}E(Z) = 0$ And the variance is

$$Var\left(\frac{3}{2}X + \frac{\sqrt{3}}{2}Z < 1\right) = \frac{9}{4}Var(X) + \frac{3}{4}Var(Z) = \frac{12}{4} = 3.$$

Thus,

$$P\left(\frac{3}{2}X + \frac{\sqrt{3}}{2}Z < 1\right) = \Phi\left(\frac{1}{\sqrt{3}}\right) = \Phi\left(\frac{\sqrt{3}}{3}\right).$$

Which is option 4.