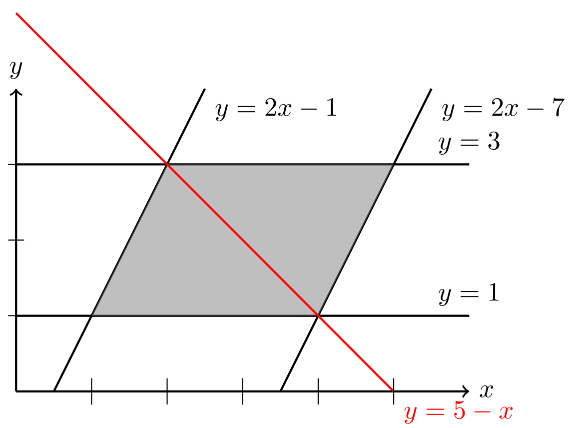
IMM - DTU

02405 Probability
2024-20-11
Hannah Nielsen

These are suggested solutions and explanations for the December 2022 exam in the course 02405 *Sandsynlighedsregning* at DTU. Page references are to the book *Probability* by Jim Pitman.

# Problem 1

Let $A$ be an area limited by $y = 2x - 1$, $y = 2x - 7$, $y = 3$ and $y = 1$




We are asked to find $P(Y < 5 - X)$ be visual inspection this can be found to $\frac{1}{2}$. Alternatively it is the fraction of A under the red line.

$$P(Y < 5 - X) = \frac{Area(Belowred)}{Area(A)} = \frac{\frac{1}{2}(4 \cdot 2)}{4 \cdot 2} = \frac{1}{2}$$

Answer 2 is correct.

# Problem 2

Let BB be the event that the break blocks need changing so that $P(B) = 0.5$ and let BD be that the break disk needs changing $P(BD|BB) = 0.4$. We need to determine $P(BB, BD)$. This can be found by use of "Multiplication Rule" on p. 37:

$$P(BB, BD) = P(BD|BB)P(BB) = 0.5 \cdot 0.4 = 0.2$$

Answer 4 is correct.

# Problem 3

We are given a positive continues stochastic variable $T$, where the hazard rate is given as

$$\lambda(t) = \lim_{h \to 0} \frac{P(T \in (t, t+h | T > t))}{h} = \frac{1}{\sqrt{t}}.$$

To find $P(T \leq \frac{1}{4})$ reference the box on p. 297.

$$\begin{aligned} P(T \leq \frac{1}{4}) &= 1 - P(T > \frac{1}{4}) \\ &= 1 - G(\frac{1}{4}) \\ &= 1 - \exp(-\int_0^{\frac{1}{4}} \frac{1}{\sqrt{u}} du) \\ &= 1 - \exp(-[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}]_0^{\frac{1}{4}}) \\ &= 1 - \exp(-(\frac{\frac{1}{2}}{\frac{1}{2}})) \\ &= 1 - \exp(-1). \end{aligned}$$

Answer 3 is correct.

# Problem 4

The situation described is exactly a Geometric distribution with $p = 1/10$ (p. 482) and $T = 5$. Therefor the probability of the success occurs in exactly the 5. try is

$$P(T = 5) = (1-p)^{5-1} p = (9/10)^4 1/10.$$

Answer 1 is correct.

# Problem 5

The probability that $X$ of the 1000 toy parts being defected can be described with a binomial distribution with $p = \frac{3}{10}$ and similarly the success with binomial and $p = \frac{7}{10}$;

$$P(X = x) = \binom{1000}{x} (\frac{7}{10})^x (\frac{3}{10})^{1000-x}$$

This can then also be extended to get the probability that the number of successes is in the interval [675;725];

$$\begin{aligned}
&P(675 \leq X \leq 725) \\
&\qquad = (\sum_{i=675}^{725} \binom{1000}{i} (\frac{7}{10})^i (\frac{3}{10})^{1000-i}.
\end{aligned}$$

We might also consider using the "Normal Approximation to the Binomial Distribution" on p. 99 since the exponent is so big. In this case, we would get;

$$\begin{aligned}
&P(675 \leq X \leq 725) \\
&\qquad \approx \Phi(\frac{725 + \frac{1}{2} - 1000\frac{7}{10}}{\sqrt{1000\frac{7}{10}\frac{3}{10}}}) - \Phi(\frac{675 - \frac{1}{2} - 1000\frac{7}{10}}{\sqrt{1000\frac{7}{10}\frac{3}{10}}}) \\
&\qquad = \Phi(\frac{25.5}{\sqrt{210}}) - \Phi(\frac{-25.5}{\sqrt{210}}).
\end{aligned}$$

Answer 1 is correct.

# Problem 6

$X_1 \sim Pois(5)$ and $Y_1 \sim Pois(3)$ then for a quarter the number of colonies appearing by Poisson random scatter, see page 228 and forwards. For the test we have $X_{1/4} \sim Pois(5/4)$ and $Y_{1/4} \sim Pois(3/4)$, and we are asked to find $P(Z_{1/4} \leq 1) = P(X_{1/4} + Y_{1/4} \leq 1)$. Then we know by "Sums of Independent Poisson Variables are Poisson" (p. 226), so $Z_{1/4} \sim Pois(5/4 + 3/4 = 2)$

$$\begin{aligned}
P(Z_{1/4} \leq 1) &= \sum_{k=0}^{1} e^{-2}(2)^k/k! \\
&= e^{-2}(2)^0/0! + e^{-2}(2)^1/1! \\
&= e^{-2}(1+2) = e^{-2}3.
\end{aligned}$$

Answer 5 is correct.

# Problem 7

We have that file size $N \sim exp(\lambda)$. 3 random files are chosen. We have 3 independent and identically distributed variables, so we can use the theorem "Density of the kth Order Statistic" on p. 326.

Then the c.d.f. and density of the exponential distributions of file size is

$$F(x) = 1 - e^{-\lambda x} \quad x \geq 0$$
$$f(x) = \lambda e^{-\lambda x} \quad x \geq 0.$$

We are looking for the density $g(x)$ of the second largest of the 3 variables, which translates to $k = 2$ and $n = 3$. Inserting all this in the formula from the theorem, we find that

$$\begin{aligned} g(x) &= nf(x)\binom{n-1}{k-1}(F(x))^{k-1}(1-F(x))^{n-k} \\ &= 3 \cdot \lambda e^{-\lambda x} \cdot \binom{3-1}{2-1}(1-e^{-\lambda x})^{2-1}(1-(1-e^{-\lambda x}))^{3-2} \\ &= 6\lambda e^{-\lambda x}(1-e^{-\lambda x})e^{-\lambda x} \\ &= 6\lambda e^{-2\lambda x} - 6\lambda e^{-3\lambda x} \end{aligned}$$

which applies whenever $x \geq 0$.

Answer 3 is correct.

# Problem 8

From p. 424 we have that the conditional is

$$P(Y_2 = y | Y_1 = 4) = \frac{P(Y_2 = y, Y_1 = 4)}{P(Y_1 = 4)}$$

and that

$$P(Y_1 = 4) = \sum_y P(Y_2 = y, Y_1 = 4) = \frac{1}{216} + \frac{1}{36} + \frac{1}{18} = \frac{19}{216}$$

Now we can find the conditional probabilities

$$\begin{aligned} P(Y_2 = 4 | Y_1 = 4) &= \frac{P(Y_2 = 4, Y_1 = 4)}{P(Y_1 = 4)} \\ &= \frac{\frac{1}{216}}{\frac{19}{216}} = \frac{1}{19} \\ P(Y_2 = 5 | Y_1 = 4) &= \frac{P(Y_2 = 4, Y_1 = 4)}{P(Y_1 = 4)} \\ &= \frac{\frac{6}{216}}{\frac{19}{216}} = \frac{6}{19} \\ P(Y_2 = 6 | Y_1 = 4) &= \frac{P(Y_2 = 4, Y_1 = 4)}{P(Y_1 = 4)} \\ &= \frac{\frac{12}{216}}{\frac{19}{216}} = \frac{12}{19}. \end{aligned}$$

Answer 3 is correct.

## Problem 9

Let $S$ denote the event that the windmill stops, and let $M$ denote the event that the wind speed is over the max value.

We can use Bayes' Theorem (p. 49). We are given the prior probabilities of a stop and not stop

$$\begin{aligned} P(S) &= 0.1 \\ P(S^{\complement}) &= 1 - 0.1 = 0.9 \end{aligned}$$

and the likelihoods of a wind speed over the max value or not a wind speed over the max value

$$\begin{aligned} P(M|S) &= 0.9 \\ P(M|S^{\complement}) &= 0.2. \end{aligned}$$

We can calculate the probability of max wind speed, by

$$\begin{aligned} P(M) &= P(M|S)P(S) + P(M|S^{\complement})P(S^{\complement}) \\ &= 0.9 \cdot 0.1 + 0.2 \cdot 0.9. \end{aligned}$$

Inserting into Bayes' formula to find the posterior probability of a stop given max wind speed, we obtain:

$$\begin{aligned} P(S|M) &= \frac{P(M|S)P(S)}{P(M)} \\ &= \frac{0.9 \cdot 0.1}{0.9 \cdot 0.1 + 0.2 \cdot 0.9} \\ &= \frac{1}{3} \end{aligned}$$

Answer 3 is correct.

## Problem 10

So since the accidents come randomly with a mean of 6 pr. year, we can assume that accident come as a Poisson arrival process and as described in the box on p. 284 the time between arrivals then follow an exponential distribution with rate 6. So denote $T \sim exp(6)$ as the time between accidents and from the memoryless property we can pose the question as

$$P(T > 2/12),$$

which we can find as

$$P(T > 2/12) = e^{-6\cdot 2/12} = e^{-1} = 0.37$$

Answer 2 is correct.

## Problem 11

The first card given has to be either ace or king and there is a $\frac{2}{13}$ chance of that, and then the second card must be the other and one card is removed from the pile giving the chance $\frac{4}{51}$ the two chances are independent of each other giving;

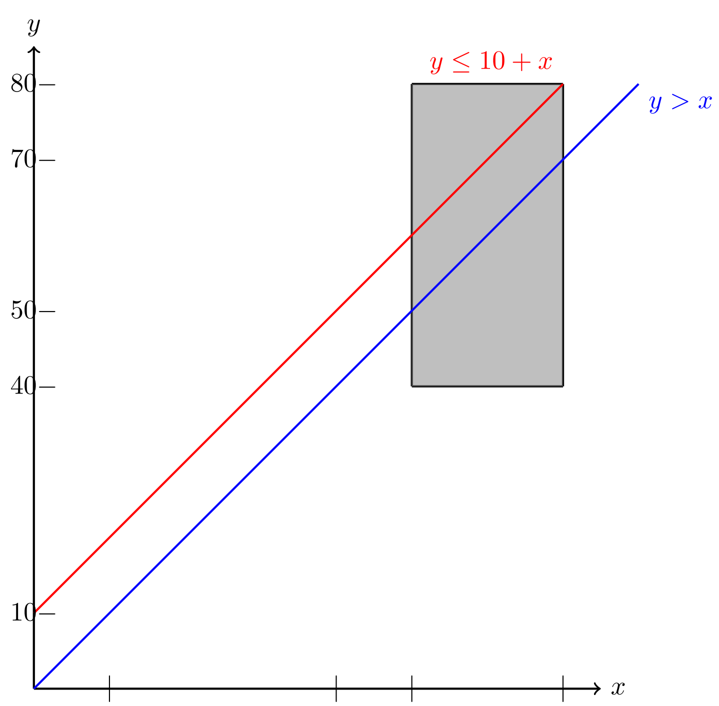
$$\frac{2}{13}\frac{4}{51} = \frac{8}{663}$$

Answer 4 is correct.

## Problem 12

Let $X$ be the arrival time of Elise and $Y$ the arrival time of Sara, where $X \sim Unif(50, 70)$ and $Y \sim Unif(40, 80)$ and $X$ and $Y$ are independent. We are interested in finding

$$P(Y \leq X + 10 | X < Y)$$



Because we are only interested in the area above the blue line since this area represent the cases where Elise arrived before Sara the area above the blue line must be $(20 \cdot 40)/2 = 400$, and then it is only the fraction of this area which is below the red line which gives us the probability of interest. Since the area above the red is more easily found as $(20 \cdot 20)/2 = 200$. The answer is

$$P(Y \leq X + 10 | X < Y) = 1 - \frac{200}{400} = 1 - \frac{1}{2} = 1/2$$

Answer 2 is correct.

## Problem 13

The random variable $Z$ is defined as the ratio of $Y$ and $X$, that is, $Z = \frac{Y}{Z}$. In this situation, we can use formula (f) on top of page 383:

$$f_Z(z) = \int_{-\infty}^{\infty} |x| f(x, zx)\, dx.$$

This formula looks simple, but we have to be quite careful. The tricky thing is to figure out *when* the joint density $f(x, zx)$ evaluates to *what*.

In this problem, we have two independent gamma(2,1) distributed variables. Hence, the joint density is $f(x, y) = (xe^{-x})(ye^{-y})$ whenever $x, y \geq 0$.

So now we can phrase the question like this: "Which values of $x$ cause the point $(x, zx) \geq (0, 0)$". This must always be true. (Note that we can assume $z = y/x$ non-negative because $X$ and $Y$ are distributed only on non-negative values.)

Using this, we can evaluate the integral:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} |x| f(x, zx)\, dx \\ &= \int_{-\infty}^{\infty} |x| \cdot x(zx) e^{-x} e^{-zx}\, dx \\ &= \int_{0}^{\infty} x^3 z e^{-x(1+z)}\, dx \\ &= \frac{6z}{(1+z)^4} \end{aligned}$$

Answer 3 is correct.

# Problem 14

Let $X$ be the number of buying customers in a store during December, $X$ is described by the density function $f_X(x) = \frac{1}{2\sqrt{x}}$, $x \in ]0;1]$. The expected value can then be found as;

$$\begin{aligned} E(X) &= \int x f_X(x)\, dx \\ &= \int_0^1 x \frac{1}{2\sqrt{x}}\, dx \\ &= \frac{1}{2}\int_0^1 \sqrt{x}\, dx \\ &= \frac{1}{2}[\frac{2}{3}\sqrt{x}^3]_0^1 \\ &= \frac{1}{3} \end{aligned}$$

Answer 3 is correct.

# Problem 15

We want to find the covariance. We want to use the alternative formula on p.430.

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 1 - 1 \cdot 1 = 0.$$

Now we find X and Y to be uncorrelated (p. 433) be keeping the warning in mind from page 430 this does not imply independence. Think if e.g. $X = 0$ is chosen then $\frac{1}{\sqrt{2}} + 1 \geq Y \geq 2$ and $Y$ can not be chosen independently of $X$.

Answer 5 is correct.

# Problem 16

We define $A_i$ as the event that the $i$th roll resulted in either 1 or 2. $X$ is a stochastic variable defined as $X = \sum_{i=1}^{8} I_{A_i}$, where $I$ is an indicator. We can then see that $X$ must follow a Binomial distribution ref p. 479.

$$\begin{aligned} P(X \leq 2) &= P(\sum_{i=1}^{8} I_{A_i} \leq 2) \\ &= P(\sum_{i=1}^{8} I_{A_i} = 0) + P(\sum_{i=1}^{8} I_{A_i} = 1) + P(\sum_{i=1}^{8} I_{A_i} = 2) \\ &= \sum_{k=0}^{2} \binom{8}{k} (\frac{1}{3})^k (\frac{2}{3})^{8-k} \end{aligned}$$

Answer 2 is correct.

# Problem 17

For the cdf of a max of n iid. see p. 319 and for cdf of the Beta with positive integer parameters see exercise 4.6.5(b) on p. 331

$$\begin{aligned}
F_{max} &= (F(x))^n \\
&= (\sum_{i=3}^{3} \binom{3}{i} x^i (1-x)^{3-i})^4 \\
&= (\binom{3}{3} x^3 (1-x)^{3-3})^4 \\
&= (x^3)^4 \\
&= x^{12} \\
&= \binom{12}{12} x^1 2 (1-x)^{12-12} \\
&= \sum_{i=12}^{12} \binom{12}{i} x^i (1-x)^{12-i}
\end{aligned}$$

We can recognize this as the cdf of a Beta(12,1) Answer 1 is correct.

# Problem 18

Let $T$ be the duration of a crisis then we are told that $E(X) = \frac{3}{2}$. Find $max(P(X \geq 4))$, to do this we can use Markov's Inequality (p. 174). The bound is then

$$max(P(X \geq 4)) = \frac{E(X)}{4} = \frac{3}{8}$$

Answer 1 is correct.

# Problem 19

Let $X_i$, $i = 1, 2$ denote the number of eyes when rolling a standard die, and define $U = \min(X_1, X_2)$ and $V = \max(X_1, X_2)$. We are asked to find $P(U + V = 4)$.

First, we list all possible outcomes of rolling two dice and calculate $U$ and $V$ for each outcome. The possible pairs $(X_1, X_2)$ that satisfy $U+V=4$ are:

$$\begin{aligned}(1,3) &\implies U=1, V=3\\ (2,2) &\implies U=2, V=2\\ (3,1) &\implies U=1, V=3\end{aligned}$$

Next, we count the number of favorable outcomes: $[(1,3),(3,1),(2,2)]$

There are 3 favorable outcomes. Since there are a total of $6\times 6=36$ possible outcomes when rolling two dice, the probability is:

$$P(U+V=4)=\frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}=\frac{3}{36}=\frac{1}{12}$$

Therefore, the probability $P(U+V=4)$ is $\frac{1}{12}$.

Answer 1 is correct.

# Problem 20

$X$ and $Y$ have standard bivariate normal distribution with correlation $\rho=\frac{1}{2}$. Then, according to "Standard Bivariate Normal Distribution" on p. 451, we can write $f_{X|Y=y}(x)$ as

$$\begin{aligned}f_{X|Y=y}(x) &\sim \mathbb{N}(\rho y, 1-\rho^2)=\mathbb{N}(\frac{y}{2},\frac{3}{4})\\ f_{X|Y=y}(x) &= \frac{1}{\sqrt{2\pi\frac{3}{4}}}e^{-\frac{1}{2}\frac{(x-\frac{y}{2})^2}{\frac{3}{4}}}\\ f_{X|Y=y}(x) &= \frac{1}{\sqrt{3/2\pi}}e^{-\frac{2}{3}(x-\frac{y}{2})^2}.\end{aligned}$$

Answer 5 is correct.

# Problem 21

To find the probability of the union use Inclusion-Exclusion from p. 22 and since A and B are independent that $P(AB)=P(A)P(B)$ (p. 42), so

$$P(A\cup B)=P(A)+P(B)-P(A)P(B)=0.2+0.15-0.2\cdot 0.15=0.32.$$

Answer 5 is correct.

## Problem 22

We have to find the probability of over 100 m from the target. To do this we can use the Rayleigh distribution (p. 358-359). However, we must adjust for the std. So the probability must be

$$\begin{aligned} P(100 < r_{50}) &= 1 - P(2 \geq r_1) \\ &= 1 - F(2) \\ &= 1 - (1 - e^{-\frac{1}{2}(2)^2}) \\ &= e^{-2} \end{aligned}$$

Answer 5 is correct.

## Problem 23

For a river we are given the water level to have distribution function:

$$F_Y(y) = P(Y \leq y) = 1 - \frac{1}{y^2},\ 1 \leq y < \infty.$$

We are asked to change variable to $Z = 10(Y-1)$. To do that we use Linear change of variable for densities from p. 333. Firstly we need the density function of $Y$.

$$f_Y(y) = 2\frac{1}{y^3},\ 1 \leq y < \infty.$$

Now we can determine the density function of $Z$.

$$f_Z(z) = \frac{1}{10}2\frac{1}{(z/10+1)^3} = \frac{1}{5}\frac{1}{(z/10+1)^3},\ 0 \leq z < \infty.$$

Lastly the distribution function can be found.

$$F_Z(z) = P(Z \leq z) = \int_0^z \frac{1}{5}\frac{1}{(u/10+1)^3} du = 1 - \frac{1}{(\frac{z}{10}+1)^2},\ 0 \leq z < \infty.$$

Answer 4 is correct.

## Problem 24

$X \sim \mathit{beta}(2,1)$ and $Y|X = x \sim \mathit{binomial}(n,x)$ We can use the *average conditional expectation* formula from the bottom of page 425:

$$E(Y) = \int E(Y|X=x) f_X(x)\, dx.$$

"The random variable $X$ is $beta(2,1)$ distributed" means (p. 327) that $X$ has density

$$f_X(x) = \frac{(2+1-1)!}{(2-1)!(1-1)!} x^{2-1}(1-x)^{1-1} = 2x.$$

And we note that this density applies only for $0 < x < 1$, as for all beta distributions. This is the limit we need to use in the integral.

"The conditional distribution of the random variable $Y$ given $X = x$ is *binamial* $(n, x)$ distributed" means, in terms of expectation, that

$$E(Y|X = x) = nx.$$

Inserting these two things in the formula above yields

$$\begin{aligned} E(Y) &= \int E(Y|X = x) f_X(x)\, dy \\ &= \int_0^1 nx2x\, dx \\ &= \frac{2n}{3}. \end{aligned}$$

Answer 1 is correct.

# Problem 25

To solve this reference p. 263. This yields;

$$P(X \in [x, x + dx]) = f(x)dx = \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{\frac{1}{2}(\log(x))^2} dx,$$

and by inserting $x$ and $dx$ we get;

$$P(X \in [1, 1 + 0.0001]) = (\frac{1}{\sqrt{2\pi}} \frac{1}{1} e^{\frac{1}{2}(\log(1))^2})(10^{-4}) = (\frac{1}{\sqrt{2\pi}} e^0)(10^{-4}) = \frac{0.0001}{\sqrt{2\pi}}.$$

Answer 3 is correct.

# Problem 26

Given $X \sim \exp(\lambda)$ and $P(Y = y|X = x) = \frac{x^y}{y!} e^{-x}$, we are asked to find the distribution of $Y$. use p. 425

$$\begin{aligned}
P(Y=y) &= \int P(Y=y|X=x)f(x)dx \\
&= \int \frac{x^y}{y!}e^{-x}\lambda e^{-\lambda x}dx \\
&= \frac{\lambda}{y!}\int x^y e^{-x(\lambda+1)}dx \\
&= \frac{\lambda}{(\lambda+1)^{y+1}} \\
&= \frac{1}{(\lambda+1)}^y \frac{\lambda}{\lambda+1} \\
&= (1-\frac{\lambda}{(\lambda+1)})^y \frac{\lambda}{\lambda+1}.
\end{aligned}$$

This we can recognize as a geometric distribution starting from 0 with parameter $\frac{\lambda}{\lambda+1}$. Answer 2 is correct.

# Problem 27

We are given that the number of $X$ (votes for party 1) and $Y$ (votes for party 2) have bivariate normal distribution with

$$\begin{aligned}
X &\sim normal(150000, 25000^2) \\
Y &\sim normal(250000, 35000^2) \\
\rho &= \frac{-1}{2}.
\end{aligned}$$

We are asked to find $P(X+Y > 450000)$.

Overall, the strategy to solve this exercise follows 3 main steps:

- Rewrite into 2 *standard* normal variables.
- Rewrite into 2 *independent* standard normal variables.
- Rewrite into 1 normal variable.

We first rewrite $X$ and $Y$ using standardized normal variables $X^*$ and $Y^*$, cf. box on p. 454:

$$\begin{aligned}
X &= \mu_X + \sigma_X X^* = 150000 + 25000X^* \\
Y &= \mu_Y + \sigma_Y Y^* = 250000 + 35000Y^*
\end{aligned}$$

The standard normal variables $X^*$ and $Y^*$ have the same correlation $\rho = \frac{-1}{2}$ as the normal variables $X$ and $Y$, according to the box on p. 454.

Using this rewrite, we have

$$\begin{aligned}P(X+Y>450000) &= P(150000+25000X^*+250000+35000Y^*>450000)\\ &= P(25000X^*+35000Y^*>50000)\\ &= P(X^*+1.4Y^*>2).\end{aligned}$$

Since $X^*$ and $Y^*$ are *standardized* bivariate normal variables, we can rewrite $Y^*$ using the formula on p. 451, with $X^*$ and $Z^*$ being *independent* standard normal variables:

$$\begin{aligned}Y^* &= \rho X^* + \sqrt{1-\rho^2}Z^*\\ &= \frac{-1}{2}\cdot X^* + \sqrt{1-\left(\frac{-1}{2}\right)^2}\cdot Z^*\\ &= \frac{-1}{2}X^* + \frac{\sqrt{3}}{2}Z^*.\end{aligned}$$

Inserting this expression, we obtain

$$\begin{aligned}P(X+Y>450000) &= P(X^*+1.4Y^*>2)\\ &= P(X^*+1.4(\frac{-1}{2}X^*+\frac{\sqrt{3}}{2}Z^*)>2)\\ &= P(0.3X^*+0.7\sqrt{3}Z^*>2).\end{aligned}$$

Now, since $X^*$ and $Z^*$ are independent standard normal variables, a linear combination $V=0.3X^*+0.7\sqrt{3}Z^*$ is a normal variable with mean zero and standard deviation given by

$$\sigma_V^2 = (\frac{3}{10})^2\cdot 1^2 + (\frac{7\sqrt{3}}{10})^2\cdot 1^2 = 1.56.$$

This is according to the formula given on p. 460 (which builds on the result for the variance of a scaling on p. 188 and the theorem about sums of independent normal variables on p. 363).

We can standardize $V$ into $V^*$ by dividing with its SD of $\sqrt{1.56}$. Doing this, we finally obtain:

$$\begin{aligned}P(X+Y>450000) &= P(X^*+1.4Y^*>2)\\ &= P(V>2)\\ &= P(V^*>\tfrac{2}{\sqrt{1.56}})\\ &= 1-\Phi(\tfrac{2}{\sqrt{1.56}})\\ &\approx 0.055.\end{aligned}$$

Answer 4 is correct.

## Problem 28

To find $P(X \leq \frac{1}{2}, Y \leq \frac{1}{2})$, we can integrate the density over this region's intersection with the triangle.

$$
\begin{aligned}
P(X \leq \frac{1}{2}, Y \leq \frac{1}{2}) &== \int_0^{\frac{1}{2}} \int_0^x f(x,y) dy dx \\
&= \int_0^{\frac{1}{2}} \int_0^x 8(x^2 - xy) dy dx \\
&= \int_0^{\frac{1}{2}} 8[(yx^2 - x\frac{1}{2}y^2)]_0^x dx \\
&= \int_0^{\frac{1}{2}} 8(x^3 - x^3\frac{1}{2}) dx \\
&= \int_0^{\frac{1}{2}} 4x^3 dx \\
&= [x^4]_0^{1/2} \\
= \frac{1}{16}
\end{aligned}
$$

Answer 4 is correct.

## Problem 29

$X$, systolic and $Y$, diastolic have standard bivariate normal distribution with correlation $\rho = \frac{5}{13}$. Then, according to the "Standard Bivariate Normal Distribution" theorem on p. 451, we can write $Y$ as

$$
\begin{aligned}
X &= \rho Y + \sqrt{1-\rho^2} Z \\
&= \tfrac{5}{13} Y + \tfrac{12}{13} Z
\end{aligned}
$$

where $X$ and $Z$ are *independent* standard normal variables.

We are asked to find the probability that diastolic bigger than 0 and systolic larger than diastolic. Written as inequalities, this is equal to the event $Y < X$ and $Y > 0$. Substituting $X = \frac{5}{13}Y + \frac{12}{13}Z$, we obtain:

$$
\begin{aligned}
P(Y < \tfrac{5}{13}Y + \tfrac{12}{13}Z \text{ and } Y > 0) &= P(\tfrac{8}{13}Y < \tfrac{12}{13}Z \text{ and } Y > 0) \\
&= P(\frac{2}{3}Y < Z \text{ and } Y > 0 \text{ and } Z > 0).
\end{aligned}
$$

As in Example 2 on p. 457, we can now use the rotational symmetry of the joint distribution of $Y$ and $Z$. (The rotational symmetry is due to the fact that $Y$ and $Z$ are *independent* standard normal variables.)

The three inequalities correspond to the region in the 1st quadrant over the line through origo with slope $\frac{2}{3}$. The angle between this line and the $Y$-axis is $Arctan(\frac{2}{3})$. Due to the rotational symmetry, the probability of landing in this region is given by this angle divided by $2\pi$, and we are then interested in the other part of this region of probability $\frac{1}{4}$, so we finally obtain the probability

$$\frac{1}{4} - \frac{Arctan(\frac{2}{3})}{2\pi} = \frac{1}{6}.$$

Answer 4 is correct.

## Problem 30

Given $X \sim \mathbb{N}(34000, 4000^2)$ and wanting to find $P(30000 < X < 35000)$ we can then transform the problem to be one with standard normal distribution by subtracting the mean and dividing with the standard deviation. So

$$\begin{aligned} P(30000 < X < 35000) &= P((30000 - 34000)/4000 < X^* < (35000 - 34000)/4000) \\ &= P(-1 < X^* < 1/4) \\ &= \Phi(1/4) - \Phi(-1) \\ &\approx 0.5987 - 0.1587 \\ &= 0.440. \end{aligned}$$

Answer 2 is correct.