IMM - DTU

02405 Probability
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BFN/bfn

# Solution for exercise 4.2.9 in Pitman

**Question a)**

$$\Gamma(r+1) = \int_0^\infty x^r e^{-x}\mathrm{d}x = \left[x^r(-e^{-x})\right]_0^\infty - \int_0^\infty r x^{r-1}(-e^{-x})\mathrm{d}x = r\int_0^\infty r x^{r-1}e^{-x}\mathrm{d}x = r\Gamma(r)$$

**Question b)** For $r = 1$ we have

$$\Gamma(1) = \int_0^\infty e^{-x}\mathrm{d}x = 1$$

and the result is proved by induction.

**Question c)**

$$E(T^n) = \int_0^\infty t^n f(t)\mathrm{d}t = \int_0^\infty t^n e^{-t}\mathrm{d}t = \Gamma(n+1)$$

$$Var(T) = E(T^2) - (E(T))^2 = \Gamma(3) - (\Gamma(2))^2 = 2 - 1 = 1$$

**Question d)** We introduce the random variable $Y = \lambda T$. The survival function of $Y$ $G_Y(y)$ can be derived through

$$G_Y(y) = P(Y > y) = P(\lambda T > y) = P\left(T > \frac{y}{\lambda}\right)$$

Now $P(T > x)e^{-\lambda x}$ such that

$$G_Y(y) = e^{-\lambda\frac{y}{\lambda}} = e^{-y}$$

the survival function of an *exponential*(1) variable. Now

$$E(T^n) = \frac{1}{\lambda^n}E((\lambda T)^n) = \frac{n!}{\lambda^n}$$

since $E((\lambda T)^n) = n!$ (the variable $Y = \lambda T$ is an *exponential*(1) distributed random variable).