IMM - DTU

02405 Probability
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BFN/bfn

Solution for exercise 3.4.12 in Pitman

We will use the formula for the geometric series

$$\sum_{i=0}^{\infty} q^i = 1 + q + q^2 \ldots = \frac{1}{1-q}, \qquad |q| < 1$$

repeatedly in this exercise.

Question a) The Rule of Averaged Conditional Probabilities p.41 applied for a countable rather than a finite partitioning. See also p.209 Infinite Sum Rule.

$$P(W_1 = W_2) = \sum_{w=1}^{\infty} P(W_1 = w) P(W_2 = W_1 | W_1 = w)$$

Now using the indpendence of W_1 and W_2 we get

$$P(W_1 = W_2) = \sum_{w=1}^{\infty} P(W_1 = w) P(W_2 = w) = \sum_{w=1}^{\infty} p_1(1-p_1)^{w-1} p_2(1-p_2)^{w-1}$$

$$= p_1 p_2 \sum_{k=0}^{\infty} ((1-p_1)(1-p_2))^k = \frac{p_1 p_2}{1-(1-p_1)(1-p_2)}$$

Question b) Similarly

$$P(W_1 < W_2) = \sum_{w=1}^{\infty} P(W_1 = w) P(W_2 > w) = \sum_{w=1}^{\infty} p_1(1-p_1)^{w-1}(1-p_2)^w = \frac{p_1(1-p_2)}{1-(1-p_1)(1-p_2)}$$

where the relation $P(W_2 > w) = (1-p_2)^w$ can be derived using the formula of the geometric series. The result is stated directly page 482.

Question c)

$$P(W_2 < W_1) = \frac{p_2(1-p_1)}{1-(1-p_1)(1-p_2)}$$

Question d)

$$P(\min(W_1, W_2) = w) = P(W_1 = W_2 = w) + P(W_1 = w, W_2 > w) + P(W_1 > w, W_2 = w)$$

$$= p_1 p_2 ((1-p_1)(1-p_2))^{w-1} + p_1(1-p_1)^{w-1}(1-p_2)^w + (1-p_1)^w p_2 (1-p_2)^{w-1}$$

$$= (p_1 p_2 + p_1(1-p_2) + (1-p_1)p_2)((1-p_1)(1-p_2))^{w-1} = (1-(1-p_1)(1-p_2))((1-p_1)(1-p_2))^{w-1}$$

We could have stated this result directly without calculations, by considering two simultaneously running series of Bernoulli experiments, where we stop as soon as we get a succes in one of them.

Question e)

$$P(\max(W_1, W_2) = w) = P(W_1 = W_2 = w) + P(W_1 = w, W_2 < w) + P(W_1 < w, W_2 = w)$$

$$= p_1 p_2((1-p_1)(1-p_2))^{w-1} + p_1(1-p_1)^{w-1}(1-(1-p_2)^{w-1}) + (1-(1-p_1)^{w-1})p_2(1-p_2)^{w-1}$$
$$= p_1(1-p_1)^{w-1} + p_2(1-p_2)^{w-1} - (p_1 + p_2 - p_1 p_2)((1-p_1)(1-p_2))^{w-1}$$