IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for exercise 6.3.8 in Pitman

Question a) Since $Y|X = x$ is $bin(5, x)$ distributed we immediately have

$$E(Y|X = x) = 5 \cdot x, E(Y^2|X = x) = 5x(1 - x) + 25x^2 = 5x(1 + 4x)$$

where we have used the computational formula for the variance to get $E(Y^2|X = x)$. Now using the boxed result page 403

$$E(Y) = E(E(Y|X)) = \int_0^1 5x \mathrm{d}x = \frac{5}{2}$$

and (once again using page 403)

$$E(Y^2) = E(E(Y^2|X)) = \int_0^1 (5x + 20x^2)\mathrm{d}x = \frac{55}{6}$$

Question b)

$$P(Y = y, x < X < x+\mathrm{d}x) = f(x)\mathrm{d}xP(Y = y|x < X < x+\mathrm{d}x) = 1\cdot\mathrm{d}x \binom{5}{y} x^y(1-x)^{5-y}$$

Question c) To find the density we consider

$$P(x < X < x + \mathrm{d}x|Y = y) = \frac{P(Y = y, x < X < x + \mathrm{d}x)}{P(Y = y)}$$

The probability $P(Y = y)$ in the denominator is found by

$$P(Y = y) = \int_0^1 1 \cdot \binom{5}{y} x^y(1 - x)^{5-y}\mathrm{d}x$$

Such that

$$P(x < X < x+\mathrm{d}x|Y = y) = f(x|Y = y)\mathrm{d}x = \frac{\binom{5}{y} x^y(1 - x)^{5-y}\mathrm{d}x}{\int_0^1 1 \cdot \binom{5}{y} x^y(1 - x)^{5-y}\mathrm{d}x}$$

which we reckognize as a $beta(y + 1, 6 - y)$ distribution (page 478).