IMM - DTU

02405 Probability
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BFN/bfn

Solution for exercise 5.3.15 in Pitman

Question a) This is exercise 4.4.10 b). We recall the result Introducing $Y = g(Z) = Z^2$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \qquad y = g(z) = z^2, \qquad z = \sqrt{y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = 2z = 2\sqrt{y}$$

Inserting in the boxed formula page 304 and use the many to one extension.

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \qquad 0 < y < \infty$$

We recognize the gamma density with scale parameter $\lambda = \frac{1}{2}$ and shape parameter $r = \frac{1}{2}$ from the distribution summary page 481. By a slight reformulation we have

$$f_Y(y) = \frac{1}{2} \frac{\left(\frac{y}{2}\right)^{\frac{1}{2}-1}}{\sqrt{\pi}} e^{-\frac{y}{2}}$$

and we deduce have

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Question b) The formula is valid for $n = 1$. Assuming the formula valid for odd n we get

$$Gamma\left(\frac{n+2}{2}\right) = \Gamma\left(\frac{n}{2} + 1\right)$$

The recursive formula for the gamma-function page 191 tells us that $\Gamma(r+1) = r\Gamma(r)$ and we derive

$$Gamma\left(\frac{n+2}{2}\right) = \frac{n}{2} \frac{\sqrt{\pi}(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}$$

$$\Gamma\left(\frac{n}{2}\right) = \prod_{i=1}^{\frac{n-1}{2}} \left(i - \frac{1}{2}\right) \sqrt{\pi}$$

Question c) Obvious by a simple change of variable.

Question d) From the additivity of the gamma distribution, which we can prove directly

Question e) From the interpretation as sums of squared normal variables.

Question f) The mean of a gamma (r, λ) distribution is $\frac{r}{\lambda}$, thus χ^n has mean $\frac{\frac{n}{2}}{\frac{1}{2}} = n$. The variance of a gamma (r, λ) distribution is $\frac{r}{\lambda^2}$, thus the variance of χ^n is $\frac{\frac{n}{2}}{\frac{1}{4}} = 2n$. Skewness bla bla bla