Solution for exercise 6.2.4 in Pitman

Question a) We first derive

$$E(Y|X=x) = \sum_{y=1}^{x} y \cdot \frac{1}{x} = \frac{1}{x}\sum_{y=1}^{x} y \quad .$$

We have the general formula (from Appendix 2 on sums page 516 (first line of last box))

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad .$$

This formula can be derived by induction a by a smart argument. For even n collect in pairs $(1,n)$, $(2,n-2)\ldots$, $(i,n+1-i)\ldots$ and realize that the sum of i and $n+1-i$ is always $n+1$ and that we have $\frac{n}{2}$ of such pairs. The extension for n odd is straightforward. with this result we get

$$E(Y|X=x) = \frac{1}{x}\sum_{y=1}^{x} y = \frac{1}{x}\frac{x(x+1)}{2} = \frac{x+1}{2} \quad .$$

Now

$$E(Y) = E(E(Y|X)) = E\left(\frac{X+1}{2}\right) = \frac{1}{2}E(X) + \frac{1}{2} = \frac{1}{2}\left(\sum_{x=1}^{n} x\frac{1}{n}\right) + \frac{1}{2}$$

$$= \frac{1}{2}\frac{1}{n}\frac{n(n+1)}{2} + \frac{1}{2} = \frac{n+3}{4}$$

Question b)

$$E(Y^2|X=x) = \sum_{y=1}^{x} y^2 \frac{1}{x}$$

We have the general formula

$$\sum_{i=1}^{m} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(which we can derive using $E(X^2) = SD(X)^2 + E(X)^2$ for the uniform distribution page 477 or 487). Thus

$$E(Y^2|X=x) = \frac{(x+1)(2x+1)}{6}$$

Now

$$E(Y^2) = E(E(Y^2|X=x)) = \sum_{x=1}^{n} \frac{(x+1)(2x+1)}{6}\frac{1}{n}$$

$$= \left(\frac{1}{3}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2} + \frac{n}{6}\right)\frac{1}{n} = \frac{(n+1)(4n+11)+6}{36}$$

Question c) To find $SD(Y)$ we use the computational formula for the variance

$$SD(Y) = \sqrt{E(Y^2) - (E(Y))^2} = \frac{\sqrt{7n^2+6n-13}}{12}$$

after simplifications.

Question d)

$$P(X+Y=2) = P(X+Y=2|X=1)P(X=1) + P(X+Y=2|X\neq 1)P(X\neq 1)$$

$$= P(X+Y=2|X=1)P(X=1) = \frac{1}{n}$$