Written examination: May 30 2016 *Course no.* : 02405

Name of course: Probability theory *Duration* : 4 hours

Aids allowed: All

The questions have been answered by:

________________________ ________________________ ________

(name) (signature) (table no.)

There are 30 exercises with a total of 30 questions. The numbering of the exercises are given as 1,2,. . . , 30 in the text. Every single question is also numbered and given as question 1,2,. . . ,30 in the text. The answers to the test must be uploaded through campusnet, using the file "answers.txt"or an equivalent file. In this file your student ID should be on the first line, the number of the question and your answer should be on the following lines, with one line for each question. The table below can be handed in as a supplement to the electronic hand-in. In case of disagreement the electronic version will be used.

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
answer															

Question	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
answer															

The options for each question are numbered from 1 to 6.

5 points are obtained for a correct answer, and -1 point for a wrong answer. Questions left unanswered or with "6" (for 'do not know') are given 0 points. The number of points needed for a sufficient exam will be determined in connection with the examination of the papers.

Notice, the idea behind the exercises is that there is one and only one correct number for every single question. All the given number options may not necessarily make sense. The last page of the exam set is page no 17; flip to that page to be sure it is there.

The notation $\log(\cdot)$ is used for the natural logarithm, i.e. logarithms with base e, while Φ denotes the cumulative distribution function for a standardised normally distributed variable.

Exercise 1

The following applies to the events D and C: $P(D) = \frac{3}{4}$, $P(C|D) = \frac{2}{9}$ and $P(C|D^c) = \frac{4}{7}$, where D^c is the complementary event of D.

Question 1

The probability $P(C)$ is

1 ☐ $\frac{1}{3}$

2 ☐ $\frac{17}{40}$

3 ☐ $\frac{13}{42}$

4 ☐ $\frac{50}{63}$

5 ☐ $\frac{41}{63}$

6 ☐ Do not know

Exercise 2

A sensor in a sensor network has two power sources. When one power source is depleted, the other automatically takes over. It is assumed that the other power source functions as new, from the moment it replaces the now depleted power source. The lifetime of the power sources can be described by an exponential distribution with mean $\frac{3}{2}$ years.

Question 2

What is the probability, that the sensor has a functioning power source after 2 years.

1 ☐ $\Phi\left(\frac{2-\frac{4}{3}}{\sqrt{\frac{8}{3}}}\right)$

2 ☐ $\left(1+\frac{8}{3}\right)e^{-\frac{8}{3}}$

3 ☐ $(1+1.6)\,e^{-1.6}$

4 ☐ $\frac{7}{3}e^{-\frac{4}{3}}$

5 ☐ $1-\Phi\left(\frac{2-\frac{4}{3}}{\sqrt{\frac{8}{3}}}\right)$

6 ☐ Do not know

Continue at page 3

Exercise 3

A random variable Y is exponentially distributed with parameter γ. Another random variable X is, given Y, normally distributed with mean $Y^2\phi$ and variance $Y^2\delta^2$.

Question 3

The mean of X is

1 ☐ $\frac{2}{\gamma}(\phi^2+\delta^2)$

2 ☐ $\frac{2}{\gamma}\phi$

3 ☐ $\frac{2}{\gamma}\phi^2$

4 ☐ $\frac{2}{\gamma^2}\phi^2$

5 ☐ $\frac{2}{\gamma^2}\phi$

6 ☐ Do not know

Exercise 4

We consider a random variable X with density $f(x)=4x^3, 0 \le x \le 1$.

Question 4

The expected value of $Y=\sqrt{X}$ is

1 ☐ $\frac{1}{2}$

2 ☐ $\frac{2\sqrt{5}}{5}$

3 ☐ $\frac{8}{9}$

4 ☐ $\frac{6}{7}$

5 ☐ $\frac{5}{6}$

6 ☐ Do not know

Continue at page 4

Exercise 5

Question 5

The survival time of a type of LED lightbulbs can be described by identically distributed, non-negative random variables X_i with a hazard rate $h(x)$, which is increasing in x.

1 ☐ It is impossible for a hazard rate to be increasing for all $x > 0$.

2 ☐ The error rate is the same for young and old LED lightbulbs.

3 ☐ Old LED lightbulbs have a smaller error rate than young LED lightbulbs, but there are more young ones.

4 ☐ Old LED lightbulbs have a larger error rate than young ones.

5 ☐ The information given is insufficient to make general statements.

6 ☐ Do not know

Exercise 6

A quarter of the individuals of a certain species of fly, possess a gene mutation. It is possible to test for this mutation. The test always gives a positive result, if the individual being tested possesses the mutation. In a quarter of the cases, where the individual being tested does not possess the gene, the test also gives a positive result. The test has been applied to an individual and given a positive result.

Question 6

On basis of the above information, the probability that the individual possess the gene mutation is

1 ☐ $\frac{3}{4}$

2 ☐ $\frac{4}{7}$

3 ☐ $\frac{1}{2}$

4 ☐ $\frac{1}{3}$

5 ☐ $\frac{1}{4}$

6 ☐ Do not know.

Continue at page 5

Exercise 7

The pair (X, Y) is bivariate standardized normally distributed with correlation coefficient $\rho = \frac{1}{2}$.

Question 7

One finds $P(X + Y \geq 0, X \geq 2Y)$ to be

1 ☐ $1 - \frac{\text{Arctan}(2)}{\pi}$

2 ☐ $\frac{\text{Arctan}(2)}{\pi}$

3 ☐ $1 - \frac{\text{Arctan}(\sqrt{5})}{\pi}$

4 ☐ $\frac{1}{3\sqrt{3}}$

5 ☐ $\frac{1}{6}$

6 ☐ Do not know

Exercise 8

Assume, that a European soccer club has a probability of 1/20 of winning the national championship, every year.

Question 8

The probability, that the club in question must wait more than 10 years to win the national championship, is

1 ☐ $\sum_{i=16}^{\infty} \begin{pmatrix} i \\ 16 \end{pmatrix} \frac{1}{20} \left(\frac{19}{20}\right)^{i-1}$

2 ☐ $\left(\frac{19}{20}\right)^{10}$

3 ☐ $\frac{1}{20^{10}}$

4 ☐ $\frac{1}{2}$

5 ☐ $\frac{1}{20} \left(\frac{19}{20}\right)^{10}$

6 ☐ Do not know

Continue at page 6

Exercise 9

A group of athletes is assumed to be on the same level in the discipline, 110-meter hurdles. Their completion times can be assumed normally distributed with mean $14s$ and variance $0,25s^2$.

Question 9

What is the probability, that the winning time is under $13,5s$ in a race with 6 runners?

1 ☐ $(1-\Phi(-1))^6$
2 ☐ $1-\Phi(-2)^6$
3 ☐ $1-(1-\Phi(-1))^6$
4 ☐ $1-(1-\Phi(-2))^6$
5 ☐ $\Phi(-2)^6$
6 ☐ Do not know

Exercise 10

An online game works in the following way. Each participant is given a number between 1 and the number of players, such that each participant has their own number. Then, a number is drawn randomly among these in each round. The winner is the one whose number is drawn first for the second time.

Question 10

The probability, that the game ends exactly in round 4, is

1 ☐ $\frac{1}{25}\left(\frac{24}{25}\right)^3$
2 ☐ $\dfrac{\binom{4}{1}\binom{21}{3}}{\binom{25}{4}}$
3 ☐ $\frac{24}{25}\cdot\frac{23}{25}\cdot\frac{3}{25}$
4 ☐ $\frac{\left(25\cdot\frac{1}{25}\right)^4}{4!}\exp\left(-\left(25\cdot\frac{1}{25}\right)\right)=\frac{e^{-1}}{4!}$
5 ☐ $\binom{4}{1}\frac{1}{25}\left(\frac{24}{25}\right)^3$
6 ☐ Do not know

Continue at page 7

Exercise 11

One considers the position of a parent bird in relation to the nest. An appropriately chosen length scale is used and a coordinate system with center at the nest. The coordinates of the birds position can then be considered as two independent standardized normally distributed variables.

Question 11

The probability, that the parent bird is more than 2 units of length away from the nest, is found to be

1 ☐ $\Phi(2)$

2 ☐ $1-\Phi(2)$

3 ☐ $\frac{1}{4}$

4 ☐ $1-e^{-1}$

5 ☐ e^{-2}

6 ☐ Do not know

Exercise 12

A tourist is gambling at a casino. The probability of winning in a single game is 0,48. The tourist is very persistent and plays a total of 1.000 games.

Question 12

The probability that the tourist wins at least 600 times is found, possibly approximately, to be

1 ☐ $\Phi\left(\frac{400+\frac{1}{2}-1000\cdot 0.52}{\sqrt{1000\cdot 0.48\cdot 0.52}}\right)$

2 ☐ $1-\Phi\left(\frac{400+\frac{1}{2}-1000\cdot 0.52}{\sqrt{1000\cdot 0.48\cdot 0.52}}\right)$

3 ☐ $\Phi\left(\frac{600+\frac{1}{2}-1000\cdot 0.48}{\sqrt{1000\cdot 0.48\cdot 0.52}}\right)$

4 ☐ $\frac{1}{2^{1000}}\sum_{i=600}^{1000}\binom{1000}{i}$

5 ☐ $\binom{1000}{600}0.48^{400}0.52^{600}$

6 ☐ Do not know

Continue at page 8

Exercise 13

The number of insects consumed by a bat on a short flight, can be described by a random variable with mean $\frac{3}{10}$ and variance $\frac{2}{5}$.

Question 13

The probability, that a bat has consumed more than 50 insects on 160 short flights is found, possibly approximately, to be

1 ☐ $1 - \Phi\left(\frac{1}{4}\right)$

2 ☐ $1 - \Phi\left(\frac{1}{32}\right)$

3 ☐ $1 - \Phi\left(\frac{50-160\cdot\frac{3}{10}+\frac{1}{2}}{\sqrt{\cdot\frac{3}{10}\frac{7}{10}}}\right)$

4 ☐ $\sum_{i=51}^{\infty}\frac{48^i}{i!}e^{-48}$

5 ☐ $\sum_{i=51}^{160}\left(\begin{array}{c}160\\ i\end{array}\right)\left(\frac{3}{10}\right)\left(\frac{7}{10}\right)^{160-i}$

6 ☐ Do not know

Exercise 14

One is given the random variables $X \sim \exp(1)$ and $Y = -X^4$.

Question 14

The density function $f_Y(y)$ for Y is

1 ☐ $f_Y(y) = e^{-\sqrt[4]{-y}}$

2 ☐ $f_Y(y) = e^{\sqrt[4]{y}}$

3 ☐ $f_Y(y) = 4|y|^3e^{y^4}$

4 ☐ $f_Y(y) = \frac{1}{4}(-y)^{-\frac{3}{4}}e^{-\sqrt[4]{-y}}$

5 ☐ $f_Y(y) = \frac{1}{4}y^{-\frac{3}{4}}e^{\sqrt[4]{y}}$

6 ☐ Do not know

Continue at page 9

Exercise 15

The immune system against measles in a group of 19 children is of interest. In the entire population, there is a probability of $\frac{2}{5}$ that a child has immunity against measles.

Question 15

The probability, that at least 10 of the 19 children are immune against measles, is

1 ☐ 0,4

2 ☐ $1 - \sum_{x=0}^{9} \frac{\left(\frac{38}{5}\right)}{x!} e^{-\frac{38}{5}}$

3 ☐ $\sum_{x=10}^{19} \left(\begin{array}{c} 19 \\ x \end{array} \right) \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{19-x}$

4 ☐ $\Phi\left(\frac{-181}{2\sqrt{114}}\right)$

5 ☐ $1 - \Phi\left(\frac{10-\frac{38}{5}-\frac{1}{2}}{\sqrt{19\cdot\frac{6}{25}}}\right)$

6 ☐ Ved ikke

Exercise 16

One has the two random variables V, W, where $Var(V) = 4$, $Var(W) = 1$ and $Var(V-2W) = 12$.

Question 16

The correlation $Corr(V, W)$ is found to be

1 ☐ $\frac{1}{2}$

2 ☐ $-\frac{1}{4}$

3 ☐ $\frac{1}{4}$

4 ☐ $-\frac{1}{2}$

5 ☐ Cannot be determined from the given information.

6 ☐ Do not know

Continue at page 10

Exercise 17

The height and weight of airplane passengers are described by a bivariate normal distribution. The average height of a passenger is 172cm, while the average weight is 65kg. The associated standard deviations are 10cm and 15kg, respectively. The correlation coefficient between height and weight is $\rho = 0.4$.

Question 17

Calculate the probability that an airplane passenger, who is 157cm tall, weighs more than 65kg.

1 ☐ 0.1843

2 ☐ 0.2563

3 ☐ 0.2967

4 ☐ 0.4294

5 ☐ 0.5000

6 ☐ Do not know

Exercise 18

One has three independent random variables $U_i, i = 1, 2, 3$, all of which are uniformly distributed in the interval $[0; 1]$.

Question 18

The probability, that the second smallest of the 3 variables is larger than $\frac{1}{4}$, is found to be

1 ☐ $\frac{55}{64}$

2 ☐ $\frac{1}{2}$

3 ☐ $\frac{57}{64}$

4 ☐ $\frac{27}{32}$

5 ☐ $\frac{3}{4}$

6 ☐ Do not know

Continue at page 11

Exercise 19

The bacteria count of hospital patients is continuously monitored. The mean bacteria count among hospital patients is 3.

Question 19

The probability, that a patient has a bacteria count of at least 60, is at most

1 ☐ 1 - $\Phi\left(\frac{60-3}{\sqrt{60}}\right)$

2 ☐ 1 - $\Phi\left(\frac{60-3}{60}\right)$

3 ☐ $\frac{1}{400}$

4 ☐ $\frac{1}{20}$

5 ☐ $\frac{1}{9}$

6 ☐ Do not know

Exercise 20

For the discrete random variables X and Y, one has the joint distribution $P(X = x, Y = y) = e^{-\lambda}\frac{(\lambda(1-p))^x}{y!(x-y)!}\left(\frac{p}{1-p}\right)^y$, where $0 \leq Y \leq X$.

Question 20

The conditional distribution of X given Y is found to be

1 ☐ $P(X = x|Y = y) = \frac{(\lambda(1-p))^x}{y!(x-y)!}\left(\frac{p}{1-p}\right)^x, \qquad x = 0, 1, \ldots$

2 ☐ $P(X = x|Y = y) = \frac{(\lambda(1-p))^{x-y}}{(x-y)!}e^{-\lambda(1-p)}, \qquad x = y, y+1, \ldots$

3 ☐ $P(X = x|Y = y) = \frac{(\lambda p))^{x-y}}{(x-y)!}e^{-\lambda p}, \qquad x = y, y+1, \ldots$

4 ☐ $P(X = x|Y = y) = \frac{(\lambda(1-p))^x}{y!(x-y)!}\left(\frac{p}{1-p}\right)^x, \qquad x = y, y+1, \ldots$

5 ☐ $P(X = x|Y = y) = \left(\begin{array}{c} x \\ y \end{array}\right) p^y(1-p)^{x-y}, \qquad$ in the domain.

6 ☐ Do not know.

Continue at page 12

Exercise 21

The size of banks' capital gap immediately preceeding a recession, and the duration of the subsequent recession, can be described by a bivariate normal distribution. The mean and variance of the capital gap can, with an appropriate scaling, be assumed to be 12 and 2^2 respectively, while the mean and variance of the duration can be assumed to be 18 and 5^2. The correlation coefficient can be assumed to be 0.6. An economist has suggested that recessions could be quantified as the sum of the capital gap and the duration.

Question 21

The probabillity, that a recession has a value smaller than 36 on the economist's scale, is found to be

1 ☐ $\Phi(6)$

2 ☐ $\Phi\left(\frac{6\sqrt{29}}{29}\right)$

3 ☐ $\Phi\left(\frac{6}{7}\right)$

4 ☐ $\Phi\left(\frac{3\sqrt{5}}{10}\right)$

5 ☐ $\Phi\left(\frac{6\sqrt{41}}{41}\right)$

6 ☐ Do not know

Exercise 22

From observations of a stretch of road it has been concluded, that the time interval between two cars can reasonably be described by an exponential distribution with mean $\frac{1}{2}$ minute.

Question 22

What is the probability, that more than three cars pass a point on the road within one minute?

1 ☐ $1 - \frac{19}{3}e^{-2}$

2 ☐ $1 - \frac{79}{48}e^{-\frac{1}{2}}$

3 ☐ $\frac{79}{48}e^{-\frac{1}{2}}$

4 ☐ $\frac{13}{2}e^{-2}$

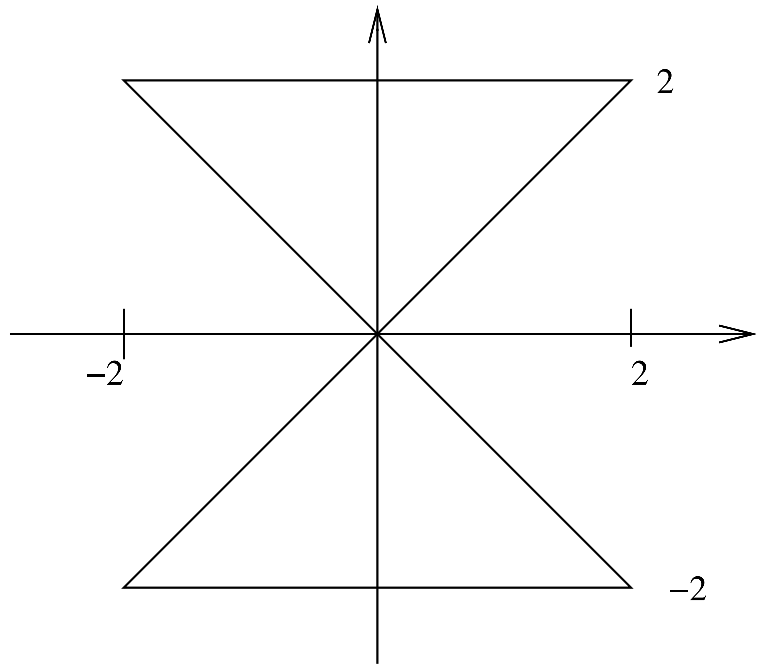
5 ☐ $1 - 5e^{-2}$

6 ☐ Do not know

Continue at page 13

Exercise 23

A point is chosen at random in the area indicated on the figure.



Question 23

The probability, that the first-coordinate X of the point is less than -1, is found to be

1 ☐ $\frac{1}{2}$

2 ☐ $\frac{1}{4}$

3 ☐ $\frac{1}{8}$

4 ☐ $\frac{3}{32}$

5 ☐ $\frac{1}{16}$

6 ☐ Do not know

Continue at page 14

Exercise 24

The random variables X and Y has joint density funcion $f(x,y) = 3e^{-x-y}$ for $0 < \frac{x}{2} \le y \le 2x$. The marginal density function is for both X and Y given by $f(x) = 3\left(e^{-\frac{3x}{2}} - e^{-3x}\right)$. Define the random variable $Z = X + Y$.

Question 24

The density $f_Z(z)$ of Z is

1 ☐ $f_Z(z) = \int_0^z 3e^{-x-(z-x)}\mathrm{d}x$

2 ☐ $f_Z(z) = \int_{\frac{z}{3}}^{\frac{2z}{3}} 3e^{-z}\mathrm{d}x$

3 ☐ $f_Z(z) = \int_{\frac{z}{3}}^{\frac{2z}{3}} 9\left(e^{-\frac{3x}{2}} - e^{-3x}\right)\left(e^{-\frac{3(z-x)}{2}} - e^{-3(z-x)}\right)\mathrm{d}x$

4 ☐ $f_Z(z) = \int_{\frac{z}{3}}^{\frac{3z}{2}} 3e^{-x-(x-z)}\mathrm{d}x$

5 ☐ $f_Z(z) = \int_{\frac{z}{3}}^{\frac{4z}{3}} 9\left(e^{-\frac{3x}{2}} - e^{-3x}\right)\left(e^{-\frac{3(z-x)}{2}} - e^{-3(z-x)}\right)\mathrm{d}x$

6 ☐ Do not know

Exercise 25

A pair of random variables (X,Y) are specified by their joint distribution function $F(x,y) = P(X \le x, Y \le y)$.

Question 25

One finds

1 ☐ $P(x_1 < X \le x_2, y_1 < Y \le y_2) = F(x_2,y_2) - F(x_1,y_1)$

2 ☐ $P(x_1 < X \le x_2, y_1 < Y \le y_2) = F(x_2,y_2) + F(x_1,y_2) + F(x_2,y_1) - F(x_1,y_1)$

3 ☐ $P(x_1 \le X \le x_2, y_1 \le Y \le y_2) = F(x_2,y_2) + F(x_1,y_2) + F(x_2,y_1) - F(x_1,y_1)$

4 ☐ $P(x_1 < X \le x_2, y_1 < Y \le y_2) = F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) + F(x_1,y_1)$

5 ☐ $P(x_1 \le X \le x_2, y_1 \le Y \le y_2) = F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) + F(x_1,y_1)$

6 ☐ Do not know

Continue at page 15

Exercise 26

A type of electronic components are produced with a varying degree of quality. The varying quality can be quantified by a random variable X, where X is uniformly distributed on the interval [1, 3]. The lifetime of a component of a given quality follows an exponential distribution with parameter X minute^{-1}.

Question 26

What is the expected time to breakdown for a randomly chosen component?

1 ☐ $\log(\sqrt{3})$ minutes

2 ☐ 2 minutes

3 ☐ $\frac{1}{2}\log(2)$ minutes

4 ☐ $\frac{9}{4}$ minutes

5 ☐ $\frac{4}{9}$ minutes

6 ☐ Do not know

Exercise 27

The time between arrivals of packages in a communication sytem can be described by an exponential distribution with parameter μ.

Question 27

The probability of observing an inter-arrival time between two packages in the interval $[t, t+\epsilon]$, where $\epsilon << \frac{1}{\mu}$, is found, possibly approximately, to be

1 ☐ $\mu e^{-\mu t}\epsilon$

2 ☐ $\Phi\left(\frac{t+\epsilon-\frac{1}{\mu}}{\frac{1}{\mu}}\right) - \Phi\left(\frac{t-\frac{1}{\mu}}{\frac{1}{\mu}}\right)$

3 ☐ $e^{-(t+\epsilon)\mu}$

4 ☐ $\lambda e^{-\lambda(t+\epsilon)}$

5 ☐ $1 - e^{-(t+\epsilon)\mu}$

6 ☐ Do not know

Continue at page 16

Exercise 28

A fair dice is thrown twice. The maximum number of dots among the two throws is denoted by X, and the sum of the number of dots in the two throws is denoted by Y. The probabilities for all cases where the total number of dots is 8 are of interest.

Question 28

The desired probabilities are found to be

1 ☐ $P(X=4, Y=8) = \frac{1}{36}, P(X=5, Y=8) = \frac{1}{18}, P(X=6, Y=8) = \frac{1}{18}$

2 ☐ $P(X=4, Y=8) = \frac{1}{18}, P(X=5, Y=8) = \frac{1}{12}, P(X=6, Y=8) = \frac{1}{18}$

3 ☐ $P(X=1, Y=8) = \frac{1}{36}, P(X=2, Y=8) = \frac{1}{18}, P(X=3, Y=8) = \frac{1}{12},$
$P(X=4, Y=8) = \frac{1}{12}, P(X=5, Y=8) = \frac{1}{18}, P(X=6, Y=8) = \frac{1}{36}$

4 ☐ $P(X=4, Y=8) = \frac{1}{18}, P(X=5, Y=8) = \frac{1}{18}, P(X=6, Y=8) = \frac{1}{18}$

5 ☐ $P(X=1, Y=8) = \frac{1}{36}, P(X=2, Y=8) = \frac{1}{36}, P(X=3, Y=8) = \frac{1}{36},$
$P(X=4, Y=8) = \frac{1}{36}, P(X=5, Y=8) = \frac{1}{36}, P(X=6, Y=8) = \frac{1}{36}$

6 ☐ Ved ikke

Exercise 29

For the two random variables (X, Y), the joint density function is given by $f(x, y) = 8xy, 0 < y < x < 1$.

Question 29

One finds

1 ☐ $\mathsf{E}(XY) = \frac{1}{2}$

2 ☐ $\mathsf{E}(XY) = \frac{8}{9}$

3 ☐ $\mathsf{E}(XY) = \frac{1}{3}$

4 ☐ $\mathsf{E}(XY) = \frac{2}{3}$

5 ☐ $\mathsf{E}(XY) = \frac{4}{9}$

6 ☐ Ved ikke

Continue at page 17

Exercise 30

One has the independent X, which follows a Poisson(2) distribution, and Y, which follows a $\left(\frac{1}{5}\right)$ distribution. One forms $Z = (2X + 3)\,(Y - 1)$.

Question 30

One finds

1 ☐ $\mathsf{E}(Z) = \sum_{i=0}^{\infty}(2i+3)(i-1)\frac{2^i}{i!}e^{-2}\left(\frac{4}{5}\right)^{i-1}\frac{1}{5^i}$

2 ☐ $\mathsf{E}(Z) = 13$

3 ☐ $\mathsf{E}(Z) = 20$

4 ☐ $\mathsf{E}(Z) = 24$

5 ☐ $\mathsf{E}(Z) = 28$

6 ☐ Do not know

End of exam