TECHNICAL UNIVERSITY OF DENMARK

Written examination: May 29 2014

Course no. : 02405

Name of course: Probability theory

Duration : 4 hours

Aids allowed: All

The questions have been answered by:

______________________ ______________________ ________

(name) (signature) (table no.)

There are 30 exercises with a total of 30 questions. The numbering of the exercises are given as 1,2,. . . , 30 in the text. Every single question is also numbered and given as question 1,2,. . . ,30 in the text. The answers to the test must be uploaded through campusnet, using the file "answers.txt"or an equivalent file. In this file your student ID should be on the first line, the number of the question and your answer should be on the following lines, with one line for each question. The table below can be handed in as a supplement to the electronic hand-in. In case of disagreement the electronic version will be used.

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
answer															

Question	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
answer															

The options for each question are numbered from 1 to 6.

5 points are obtained for a correct answer, and -1 point for a wrong answer. Questions left unanswered or with "6" (for 'do not know') are given 0 points. The number of points needed for a sufficient exam will be determined in connection with the examination of the papers.

Notice, the idea behind the exercises is that there is one and only one correct number for every single question. All the given number options may not necessarily make sense. The last page of the exam set is page no 17; flip to that page to be sure it is there.

The notation $\log(\cdot)$ is used for the natural logarithm, i.e. logarithms with base e, while Φ denotes the cumulative distribution function for a standardised normally distributed variable.

Exercise 1

A commissions paid telemarketer calls 20 randomly selected people during her duty. Each customer accepts the offer with a probability of 10%.

Question 1

The probability that at most two customers accepts the offer during the telemarketers duty is

1 ☐ 0.1

2 ☐ 0.5

3 ☐ $\Phi\left(\frac{\frac{1}{2}}{\sqrt{20 \cdot 0.1 \cdot 0.9}}\right)$

4 ☐ $5e^{-2}$

5 ☐ $\left(\frac{9}{10}\right)^{20} + 20\frac{1}{10}\left(\frac{9}{10}\right)^{19} + 190\left(\frac{1}{10}\right)^{2}\left(\frac{9}{10}\right)^{18}$

6 ☐ Do not know

Exercise 2

A person in a damp forest receives on the average one insect bite every 5 minutes. It is assumed that the insects bite independently of each other.

Question 2

The number of bites a person receives during half an hour is best described by a

1 ☐ Poisson(6) distribution

2 ☐ geometric$\left(\frac{1}{6}\right)$ distribution

3 ☐ binomial$\left(12, \frac{1}{2}\right)$ distribution

4 ☐ uniform$\left(12, \frac{1}{2}\right)$ distribution

5 ☐ negative binomial$\left(12, \frac{1}{2}\right)$ distribution

6 ☐ Do not know

Continue at page 3

Exercise 3

Two instruments in a cockpit can replace each other. The probability that the first one fails is p, while the probability that both fail is q. We consider a case where the first instrument has failed.

Question 3

What is the probability that the second instrument fails too

1 ☐ p

2 ☐ $\frac{q}{p}$

3 ☐ $p - q$

4 ☐ $\sqrt{q}$

5 ☐ qp

6 ☐ Do not know

Exercise 4

The random variable X follows a Poisson(2) distribution, while the random variable Y follows a hypergeometric(10,100,20) distribution.

Question 4

One gets

1 ☐ $\mathsf{E}(X+Y) = 2 + \frac{100!}{90!}\frac{10!}{20!}$

2 ☐ $\mathsf{E}(X+Y) = \frac{19}{5}$

3 ☐ $\mathsf{E}(X+Y) = 4$

4 ☐ $\mathsf{E}(X+Y) = \frac{17}{5}$

5 ☐ $\mathsf{E}(X+Y)$ can not be determined as the Poisson distribution and the hypergeometric distribution are incompatible

6 ☐ Do not know

Continue at page 4

Exercise 5

It is given that $\mathsf{E}(X) = 2$, $\mathsf{E}(X^2) = 4$, $\mathsf{E}(Y) = 0$, $\mathsf{E}(Y^2) = 9$, and $Z = X + Y$.

Question 5

One gets

1 ☐ $\mathsf{Var}(Z) = 7$

2 ☐ $\mathsf{Var}(Z) = 9$

3 ☐ $\mathsf{Var}(Z) = 11$

4 ☐ $\mathsf{Var}(Z) = 13$

5 ☐ The question can not be answered due to lack of information.

6 ☐ Do not know

Exercise 6

A probability model has been developed to describe the occurrence of two species of kangaroos. There is a sufficient number of individuals to make a bivariate normal distribution an appropriate model. The expected number of each species is 800, the variance for both species is 1600, while the correlation coefficient is $\rho = -\frac{2}{5}$. A number of 700 individuals has been counted for one of the species.

Question 6

The expected number of the second (yet not counted) species is

1 ☐ 735

2 ☐ 760

3 ☐ 800

4 ☐ 840

5 ☐ 865

6 ☐ Do not know

Continue at page 5

Exercise 7

The continuous random variable V is uniformly distributed on the unit interval(0,1). One defines $X = e^V$.

Question 7

The density $f_X(x)$ for X is

1 ☐ $f_X(x) = e^{-x}, \qquad x > 0$

2 ☐ $f_X(x) = \frac{1}{e}, \qquad 0 < x < e$

3 ☐ $f_X(x) = \frac{1}{x}, \qquad 1 < x < e$

4 ☐ $f_X(x) = \frac{1}{e-1}, \qquad 1 < x < e$

5 ☐ $f_X(x) = \log(x), \qquad 1 < x < e$

6 ☐ Do not know

Exercise 8

The life time of an electronic component can be described by a gamma(3,1) distribution, where the time unit is one year. The component has reached an age of 2 years.

Question 8

The probability that the component fails within one week after it has achieved the age of two years can, possibly approximately, be calculated as

1 ☐ $4e^{-2}\frac{1}{52}$

2 ☐ $\int_2^{2+\frac{1}{52}} \frac{x^2}{2} e^{-x} \mathrm{d}x$

3 ☐ $e^{-2}\frac{1}{52}$

4 ☐ $\frac{\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{3}}e^{-\frac{1}{6}}}{1-\Phi\left(-\frac{\sqrt{3}}{3}\right)}\frac{1}{52}$

5 ☐ $\frac{1}{130}$

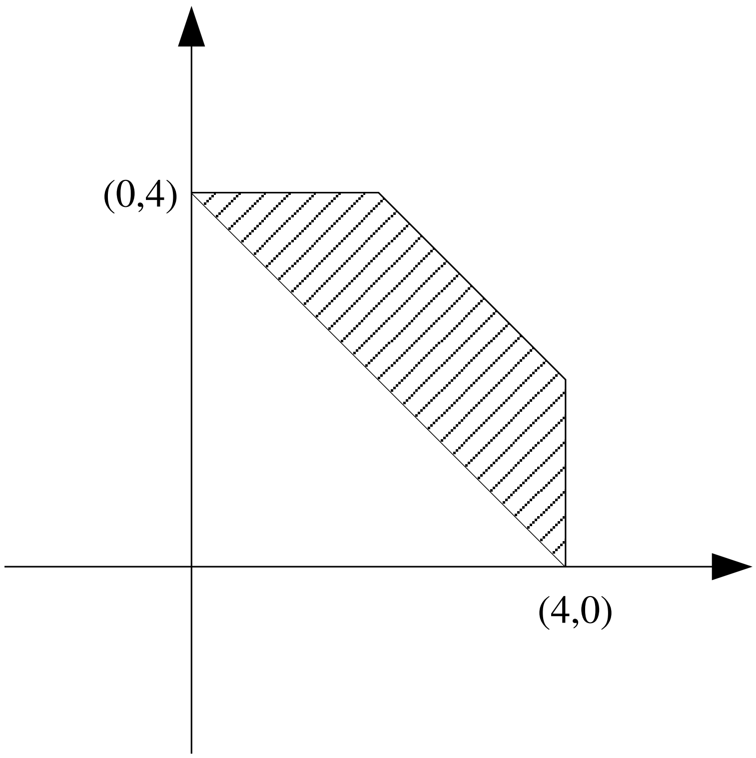
6 ☐ Do not know

Continue at page 6

Exercise 9

A point is randomly chosen in the area confined by the lines $x = 0, y = 4, x + y = 6, x = 4$, and $y = 0$.

The shaded area in the figure below is confined by the lines $y = 4, x + y = 6, x = 4$, and $x + y = 4$.



Question 9

The probability that the point is in the shaded area is

1 ☐ $\frac{5}{8}$

2 ☐ $\sqrt{2} - 1$

3 ☐ $\frac{\sqrt{3}-1}{2}$

4 ☐ $\frac{5}{9}$

5 ☐ $\frac{3}{7}$

6 ☐ Do not know

Continue at page 7

Exercise 10

Question 10

The probability that the second largest of 3 independent uniform(0,1) distributed random variables is between $\frac{1}{4}$ and $\frac{3}{4}$ is

1 ☐ $\frac{15}{32}$

2 ☐ $\frac{1}{2}$

3 ☐ $\frac{11}{16}$

4 ☐ $\frac{3}{4}$

5 ☐ $\frac{21}{32}$

6 ☐ Do not know

Exercise 11

Middle school students are classified according to their abilities within reading and arithmetics, 42% read satisfactorily, 35% have satisfactory arithmetic skills, while 55% have satisfactory skills in at least one of the disciplines.

Question 11

The percentage of students that have satisfactory skills within both reading and arithmetics is

1 ☐ 7%

2 ☐ 13%

3 ☐ 14.7%

4 ☐ 22%

5 ☐ 35%

6 ☐ Do not know

Continue at page 8

Exercise 12

It has been empirically established, that a person, that practises one of the sports cycling (C), football (F), and swimming (S) has an elevated value of hematocrit with probability P_{CH}, P_{FH}, and P_{SH} respectively. The probability that a person practises cycling, football, or swimming is P_C, P_F, and P_S respectively. It can be assumed that a person practices exactly one of the three sports mentioned above. A person has an elevated value of hematocrit.

Question 12

The probability that the person practises football is

1 ☐ $P_F P_{FH}$

2 ☐ P_{FH}

3 ☐ $1 - P_C - P_S + P_{CH} + P_{FH} + P_{SH}$

4 ☐ $\frac{P_F P_{FH}}{P_C P_{CH} + P_F P_{FH} + P_S P_{SH}}$

5 ☐ $\frac{P_{FH}}{P_F (P_{CH} + P_{FH} + P_{SH})}$

6 ☐ Do not know

Exercise 13

From a standard deck of cards 5 cards are drawn randomly.

Question 13

The probability, that among the 5 cards there are exactly 3 red and 2 black cards, is

1 ☐ $\frac{26 \cdot 25 \cdot 24 \cdot 26 \cdot 25}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$

2 ☐ $\frac{13}{40}$

3 ☐ $\binom{5}{3} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3$

4 ☐ $\frac{\binom{26}{3}\binom{26}{2}}{\binom{52}{5}}$

5 ☐ $\binom{4}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3$

6 ☐ Do not know

Continue at page 9

Exercise 14

One considers two independent random variables X and Y both following a geometric distribution with parameter p and q respectively.

Question 14

One calculates $\mathsf{P}(X > Y)$ by

1 ☐ $\mathsf{P}(X > Y) = \frac{q(1-p)}{p+q-pq}$

2 ☐ $\mathsf{P}(X > Y) = \frac{q}{3p}$

3 ☐ $\mathsf{P}(X > Y) = \frac{(1-p)(1-q)}{1-pq}$

4 ☐ $\mathsf{P}(X > Y) = \frac{p(1-q)}{p+q-pq}$

5 ☐ $\mathsf{P}(X > Y) = \frac{(1-p)q}{1-pq}$

6 ☐ Do not know

Exercise 15

The random variable X has density $2x$ for $0 < x < 1$ and 0 otherwise.

Question 15

One gets

1 ☐ $\mathsf{P}(\frac{1}{4} < X < \frac{1}{2}) = \frac{1}{4}$

2 ☐ $\mathsf{P}(\frac{1}{4} < X < \frac{1}{2}) = \frac{1}{2}$

3 ☐ $\mathsf{P}(\frac{1}{4} < X < \frac{1}{2}) = \frac{3}{16}$

4 ☐ $\mathsf{P}(\frac{1}{4} < X < \frac{1}{2}) = \frac{5}{8}$

5 ☐ $\mathsf{P}(\frac{1}{4} < X < \frac{1}{2}) = \frac{3}{8}$

6 ☐ Do not know

Continue at page 10

Exercise 16

Individuals can transfer pictures to a homepage. The size of the pictures can be described by a distribution with density $f(x) = \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}$, where α and β are positive constants.

Question 16

The probability that the smallest of the 4 pictures has a size exceeding a value of x_0 is

1 ☐ $\left(1+\frac{x_0}{\beta}\right)^{-4\alpha}$

2 ☐ $\frac{\alpha}{\beta}\left(1+\frac{x_0}{\beta}\right)^{-4\alpha-4}$

3 ☐ $\left(1-\left(1+\frac{x_0}{\beta}\right)^{-\alpha}\right)^4$

4 ☐ $\left(1-\frac{\alpha}{\beta}\left(1+\frac{x_0}{\beta}\right)^{-\alpha-1}\right)^4$

5 ☐ $\left(1+\frac{4x_0}{\beta}\right)^{-\alpha}$

6 ☐ Do not know

Exercise 17

The random variable Y is for given $X = x$ uniformly distributed on the interval $(x, 1)$, while X has density $f_X(x) = 2 - 2x, 0 \le x \le 1$.

Question 17

One calculates $\mathsf{E}(Y)$ to be

1 ☐ $\mathsf{E}(Y) = \frac{2}{3}$

2 ☐ $\mathsf{E}(Y) = \frac{1}{2}$

3 ☐ $\mathsf{E}(Y) = \frac{3}{4}$

4 ☐ $\mathsf{E}(Y) = \frac{3}{5}$

5 ☐ $\mathsf{E}(Y) = \frac{x+1}{2}$

6 ☐ Do not know

Continue at page 11

Exercise 18

Visitors in a theme park throw balls against a centered disc. The point of impact for each ball can be described by the coordinates (X_1, X_2), being the horizontal respectively the vertical distance from the center. It is reasonable to model X_1 and X_2 as two independent standard normal random variables. To win a price one has to hit a ring with a width of 1. The inner diameter of the ring is 2 (and thus the outer diameter is 4).

Question 18

The probability that a visitor wins exactly two prices by two throws is

1 ☐ $e^{-1} + e^{-4} - 2e^{-\frac{5}{2}}$

2 ☐ $e^{-2}\left(1 + e^{-2} - 2e^{-1}\right)$

3 ☐ $\frac{3}{4}$

4 ☐ $\frac{2\text{Atan}(3)}{\pi}$

5 ☐ $e^{-1} - e^{-4}$

6 ☐ Do not know

Exercise 19

A pig transporter contains 400 slaughter pigs. The slaughter pigs have a mean weight of 97.5 kg with a standard deviation of 15 kg.

Question 19

The probability that the total weight of the slaughter pigs in a pig transporter exceeds 40 tonnes can, possibly approximately, be calculated to

1 ☐ $1 - \Phi\left(\frac{20-400\cdot 97.5-\frac{1}{2}}{2000\sqrt{0.15\cdot 0.85}}\right)$

2 ☐ $1 - \Phi\left(\frac{40\ 000-400\cdot 97.5}{30\ 000}\right)$

3 ☐ $\Phi(-3)$

4 ☐ $\Phi\left(-\frac{10}{3}\right)$

5 ☐ The probability can not be calculated due to insufficient information

6 ☐ Do not know

Continue at page 12

Exercise 20

The point (X, Y) is chosen uniformly distributed within the area confined by the axes of the coordinate system and the line $2y + x = 2$. One considers $Z = X + Y$.

Question 20

The density $f_Z(z)$ for Z is

1 ☐ $f_Z(z) = \frac{z}{9}, \qquad 0 < z < 3$

2 ☐ $f_Z(z) = \begin{cases} \frac{z}{2} & 0 < z \leq 1 \\ \frac{1}{2} & 1 < z \leq 2 \\ \frac{3-z}{2} & 2 < z \leq 3 \end{cases}$

3 ☐ $f_Z(z) = \begin{cases} z & 0 \leq z < 1 \\ 2 - z & 1 \leq z \leq 2 \end{cases}$

4 ☐ $f_Z(z) = \int_0^z \frac{1}{2} \mathrm{d}x$

5 ☐ $f_Z(z) = \begin{cases} z & 0 \leq z < 1 \\ 4 - 2z & 1 \leq z \leq 2 \end{cases}$

6 ☐ Do not know

Exercise 21

Consider the pair (X, Y), which is standard bivariate normally distributed with correlation coefficient $\rho = \frac{3}{5}$, and let $Z = X + Y$.

Question 21

One calculates $\mathsf{P}(Z < -1)$ to

1 ☐ $\Phi\left(-\frac{\sqrt{5}}{5}\right)$

2 ☐ $\Phi\left(-\frac{\sqrt{2}}{2}\right)$

3 ☐ $\Phi\left(-\frac{4\sqrt{5}}{5}\right)$

4 ☐ $\Phi\left(-\frac{4\sqrt{3}}{5}\right)$

5 ☐ $\Phi\left(-\frac{\sqrt{5}}{4}\right)$

6 ☐ Do not know

Continue at page 13

Exercise 22

Airplanes are under special weather conditions redirected to a minor airport with safe conditions for landing. It is assumed that the redirected airplanes arrive independently with a frequency of 1 every third minute.

Question 22

The probability that more than 6 airplanes arrive within 5 minutes can be calculated to

1 ☐ $1 - \Phi\left(\frac{6-\frac{5}{3}}{\sqrt{\frac{5}{3}}}\right)$

2 ☐ 0.0467

3 ☐ 0.023

4 ☐ 0.017

5 ☐ 0.0017

6 ☐ Do not know

Exercise 23

The random variables X and Y have the joint density $f(x,y) = x + y$ for $0 < x < 1$ and $0 < y < 1$.

Question 23

One calculates $\mathsf{P}\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right)$ to

1 ☐ $\mathsf{P}\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right) = \frac{1}{2}$

2 ☐ $\mathsf{P}\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right) = \frac{1}{4}$

3 ☐ $\mathsf{P}\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right) = \frac{1}{6}$

4 ☐ $\mathsf{P}\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right) = \frac{1}{8}$

5 ☐ $\mathsf{P}\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right) = \frac{1}{16}$

6 ☐ Do not know

Continue at page 14

Exercise 24

One considers the events A and B and the related indicator functions I_A and I_B.

Question 24

One determines $\mathsf{P}(I_A I_B = 1)$ to be

1 ☐ $\mathsf{P}(A \cap B)$

2 ☐ $\mathsf{P}(A \cup B)$

3 ☐ $\mathsf{P}(A) + \mathsf{P}(B)$

4 ☐ $\mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$

5 ☐ The probability can not be determined without closer knowledge of the functions I_A and I_B.

6 ☐ Do not know

Exercise 25

One considers the three independent random variables X_1, X_2, and X_3, which all follow an exponential(λ) distribution. One introduces $Y = X_1 + X_2 + X_3$.

Question 25

One calculates $\mathsf{P}(Y > y)$ to be

1 ☐ $e^{-\lambda y/3}$

2 ☐ $\left(1 + \lambda y + \frac{1}{2}(\lambda y)^2\right) e^{-\lambda y}$

3 ☐ $\Phi\left(\frac{y - \frac{3}{\lambda}}{\frac{\sqrt{3}}{\lambda}}\right)$

4 ☐ $1 - \Phi\left(\frac{y - \frac{3}{\lambda}}{\frac{\sqrt{3}}{\lambda}}\right)$

5 ☐ $\lambda \frac{1}{2}(\lambda y)^2 e^{-\lambda y}$

6 ☐ Do not know

Continue at page 15

Exercise 26

A steel mill produces two different kinds of steel. The deviations from the normal production on any given day can be described using a bivariate normal distribution with correlation coefficient $-\frac{\sqrt{2}}{2}$.

Question 26

The probability that less than the normal production is produced for both kinds of steel on a given day is

1 ☐ $\frac{\frac{\pi}{2}-\mathrm{Atan}\left(\frac{\sqrt{2}}{2}\right)}{2\pi}$

2 ☐ $\frac{\mathrm{Atan}\left(\frac{\sqrt{2}}{2}\right)}{2\pi}$

3 ☐ $\frac{1}{8}$

4 ☐ $\frac{1}{6}$

5 ☐ $\frac{1}{4}$

6 ☐ Do not know

Exercise 27

Each day a company sells goods in an environment that can be described by a continuous random variable X, that takes a value in the interval $(0,1)$ with density $6x(1-x)$ independently from day to day. The number of units that can be sold during a day can be described by a binomial$(20,x)$ distribution for given x.

Question 27

The probability that exactly 10 units can be sold on an arbitrary day is

1 ☐ $\frac{1}{21}$

2 ☐ $\frac{11}{161}$

3 ☐ $\int_0^1 \binom{20}{10} 6x(1-x)\mathrm{d}x$

4 ☐ $\binom{20}{10}\frac{1}{1024}$

5 ☐ $\int_0^1 6x^{11}(1-x)^{11}\mathrm{d}x$

6 ☐ Do not know

Continue at page 16

Exercise 28

The parameter θ is used in an alternative parameterisation of the binomial distribution. The relation between θ and the usual probability parameter p is given by

$$p = \frac{e^{\theta}}{1+e^{\theta}}, \qquad \theta = \log\left(\frac{p}{1-p}\right).$$

Question 28

Let X be binomially distributed with counting parameter $n = 2$; for $\theta = 0$ one gets

1 ☐ $\mathsf{P}(X = 0) = 1$

2 ☐ $\mathsf{P}(X = 0) = \frac{1}{2}$

3 ☐ $\mathsf{P}(X = 0) = \frac{1}{4}$

4 ☐ $\mathsf{P}(X = 0) = 0$

5 ☐ $\mathsf{P}(X = 0)$ can not be determined as $\theta = 0$ is an invalid parameter value

6 ☐ Do not know

Exercise 29

The travel distance to a research station in the Arctic has a mean of 1day and a standard deviation of 3 days.

Question 29

A best upper limit for the probability that the travel time to the research station exceeds two days is

1 ☐ 1

2 ☐ $\frac{1}{2}$

3 ☐ $1 - \Phi\left(\frac{1}{3}\right)$

4 ☐ $\frac{1}{4}$

5 ☐ Can not be determined without more knowledge of the distribution of the travel time

6 ☐ Do not know

Continue at page 17

Exercise 30

The joint distribution of the discrete random variables X and Y is given by the following table.

		Values of X 1	2	3	Distribution of Y
Values of Y	3	1/6	0	1/6	1/3
	2	0	1/3	0	1/3
	1	1/6	0	1/6	1/3
	Distribution of X	1/3	1/3	1/3	1 (total sum)

Question 30

The following can be stated about the random variables X and Y

1 ☐ They are independent

2 ☐ They are uncorrelated but not independent

3 ☐ They are uncorrelated

4 ☐ They are identical

5 ☐ They are mutually exclusive

6 ☐ Do not know

End of exam