Written examination: May 28 2014 *Course no.* : 02405

Name of course: Probability theory *Duration* : 4 hours

Aids allowed: All

The questions have been answered by:

________________________ ________________________ ________

(name) (signature) (table no.)

There are 30 exercises with a total of 30 questions. The numbering of the exercises are given as 1,2,. . . , 30 in the text. Every single question is also numbered and given as question 1,2,. . . ,30 in the text. The answers to the test must be uploaded through campusnet, using the file "answers.txt"or an equivalent file. In this file your student ID should be on the first line, the number of the question and your answer should be on the following lines, with one line for each question. The table below can potentially be handed in as a supplement to the electronic hand-in. In case of disagreement the electronic version will be used.

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
answer															

Question	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
answer															

The options for each question are numbered from 1 to 6. If a wrong number has been given, it can be corrected by "blackening"out the wrong answer and writing the correct number below. In case of doubt about a correction, the question will be considered unanswered.

Draft and intermediate calculations will **not** be taken into account. Only the numbers written in the table above will be scored.

5 points are obtained for a correct answer, and -1 point for a wrong answer. Questions left unanswered or with “6” (for 'do not know') are given 0 points. The number of points needed for a sufficient exam will be determined in connection with the examination of the papers.

Notice, the idea behind the exercises is that there is one and only one correct number for every single question. All the given number options may not necessarily make sense. The last page of the exam set is page no 16; flip to that page to be sure it is there.

The notation $\log(\cdot)$ is used for the natural logarithm, i.e. logarithms with base e, while Φ denotes the cumulative distribution function for a standardised normally distributed variable.

Exercise 1

A car has a problem with its oil-pressure sensor. If the oil warning is on there is a probability of 1/5 of there being a problem with the oil pressure. The probability of the oil warning being on is 1/15 on a given trip.

Question 1

The probability that the oil warning is on, and there being problems with the oil pressure on a a given trip is

1 ☐ $\frac{2}{15}$

2 ☐ $\frac{4}{15}$

3 ☐ $\frac{1}{75}$

4 ☐ $\frac{1}{5}$

5 ☐ $\frac{1}{15}$

6 ☐ Do not know

Exercise 2

At a single point in a pneumatic post system the owners of the system are experimenting with shooting out an item to hit inside a funnel. If we impose a coordinate system, such that the centre of the coordinate system corresponds to the centre of the funnel we may describe the coordinates of the items impact site by independent standard normally distributed variables.

Question 2

The probability that an item lands at a distance more than 3 from the centre of the funnel is

1 ☐ $e^{-9/2}$

2 ☐ $\left(1 - \Phi\left(\frac{3\sqrt{2}}{2}\right)\right)^2$

3 ☐ $4e^{-3}$

4 ☐ $1 - \Phi\left(\frac{3\sqrt{2}}{2}\right)$

5 ☐ e^{-3}

6 ☐ Do not know

Continue at page 3

Exercise 3

10 deer are caught in a specific area tagged and released back into the wild. After a small interval of time, 8 deer are caught. It may be assumed that the population of deer remains constant in this time period, and that tagged and untagged deer have the same probability of being caught in the second catch. The unknown population size is denoted by n.

Question 3

The probability of catching exactly 5 tagged deer the second time may be expressed as

1 ☐ $\binom{8}{5}\left(\frac{10}{n}\right)^5\left(\frac{n-10}{n}\right)^3$

2 ☐ $\frac{8\cdot7\cdot6\cdot5\cdot4}{n(n-1)(n-2)(n-3)(n-4)}$

3 ☐ $\binom{8}{5}\frac{10\cdot9\cdot8\cdot7\cdot6}{n(n-1)(n-2)(n-3)(n-4)}\frac{(n-10)(n-11)(n-12)}{n(n-1)(n-2)}$

4 ☐ $\frac{\binom{10}{5}\binom{n-10}{3}}{\binom{n}{8}}$

5 ☐ $\binom{7}{4}\left(\frac{10}{n}\right)^5\left(\frac{n-10}{n}\right)^3$

6 ☐ Do not know

Exercise 4

A predatory fish meets prey randomly, averaging 6 hours between each encounter.

Question 4

The probability the predatory fish must wait more than 24 hours to encounter 4 of its prey is

1 ☐ e^{-4}

2 ☐ $\frac{71}{3}e^{-4}$

3 ☐ $\frac{23}{3}e^{-2}$

4 ☐ $\frac{103}{3}e^{-4}$

5 ☐ $\frac{1}{2}$

6 ☐ Do not know

Continue at page 4

Exercise 5

A fish larva has consumed 100 of its prey, which follow a distribution with mean 1.5 nanograms and standard deviation 5 nanograms.

Question 5

The probability that the larva has consumed at most 300 nanograms is, possibly approximately, found to be

1 ☐ $\Phi\left(\frac{200+\frac{1}{2}-100\cdot 1{,}5}{\sqrt{100\frac{1{,}5}{5}\frac{3{,}5}{5}}}\right)$

2 ☐ $\Phi\left(\frac{3}{2}\right)$

3 ☐ $1-\Phi\left(\frac{3}{2}\right)$

4 ☐ $\Phi(3)$

5 ☐ The probability cannot be calculated due to insufficient information

6 ☐ Do not know

Exercise 6

An air route has seats on both economy and business class. The probability of at least one of the classes being sold out is 1/3, and the probability of there being sold out on business class is 1/8.

Question 6

The probability of there being sold out on economy but not business class is

1 ☐ $\frac{1}{24}$

2 ☐ $\frac{5}{24}$

3 ☐ $\frac{9}{24}$

4 ☐ $\frac{1}{6}$

5 ☐ The probability cannot be calculated due to insufficient information

6 ☐ Do not know

Continue at page 5

Exercise 7

Let U be an $(0,1)$ uniformly distributed continuous random variable, and Y be a continuous random variable. For a given $U = u$, Y follows an exponential distribution with intensity u.

Question 7

The survival function $G_Y(y) = \mathsf{P}(Y > y)$ is

1 ☐ $G_Y(y) = e^{u/2}$

2 ☐ $G_Y(y) = \frac{1}{y}\left(1 - e^{-y}\right)$

3 ☐ $G_Y(y) = (1 + \frac{y}{2})e^{-\frac{y}{2}}$

4 ☐ $G_Y(y) = e^{-y/2}$

5 ☐ $G_Y(y) = \int_0^1 \left(1 - e^{-y}\right) \cdot 1\mathrm{d}y$

6 ☐ Do not know

Exercise 8

An injured person performs a call to an emergency hotline. However, due to heavy load caused by the weather conditions there is a probability of $9/10$ that a call will fail.

Question 8

The probability that more than 5 calls are necessary before coming into contact with the hotline is

1 ☐ 0,0591

2 ☐ 0,4066

3 ☐ 0,5905

4 ☐ 0,6065

5 ☐ 0,6321

6 ☐ Do not know

Continue at page 6

Exercise 9

The positive random variable X with density $f_X(x) = xe^{-x}$ is transformed to the random variable $Y = e^X - 1$.

Question 9

The density $f_Y(y)$ of Y is

1 ☐ $f_Y(y) = \frac{1}{(y+1)^2}$

2 ☐ $f_Y(y) = \log(y+1)e^{-(y+1)}$

3 ☐ $f_Y(y) = \frac{\log(y+1)}{4(y+1)^3}$

4 ☐ $f_Y(y) = \frac{\log(y+1)}{(y+1)^2}$

5 ☐ $f_Y(y) = \frac{w\log(y+1)}{y+1}$

6 ☐ Do not know

Exercise 10

The pair of random variables (X, Y) follows a standard bivariate normal distribution with correlation coefficient $\rho = -7/8$.

Question 10

We find $\mathsf{P}(X + Y \leq 1)$ to be

1 ☐ $\Phi(2)$

2 ☐ $\Phi\left(\frac{\sqrt{2}}{2}\right)$

3 ☐ $\Phi\left(2\sqrt{2}\right)$

4 ☐ $\Phi\left(\sqrt{2}\right)$

5 ☐ $\Phi\left(\frac{1}{2}\right)^2$

6 ☐ Do not know

Continue at page 7

Exercise 11

A lotto player buys a row every week. The probability of getting a pay-out on the row in a given week is 1/5.

Question 11

The probability of getting a pay-out in at least 10 out of 100 weeks is

1 ☐ $\Phi\left(\frac{10-20+\frac{1}{2}}{\sqrt{100\frac{4}{25}}}\right)$

2 ☐ $\sum_{i=10}^{100} \frac{20^i}{i!} e^{-20}$

3 ☐ $1 - 5^{-100} \sum_{i=0}^{9} \binom{100}{i} 4^{100-i}$

4 ☐ $\frac{\binom{20}{10}\binom{80}{10}}{\binom{100}{20}}$

5 ☐ $\sum_{i=10}^{100} \binom{100}{i} \frac{1}{2^{100}}$

6 ☐ Do not know

Exercise 12

Let X, Y be independent continuous random variables, both following a uniform$(0,1)$ distribution.

Question 12

We calculate $\mathsf{P}\left(X + Y \leq \frac{1}{2}\right)$ to be

1 ☐ $\frac{1}{6}$

2 ☐ $\frac{1}{3}$

3 ☐ $\frac{1}{2}$

4 ☐ $\frac{1}{4}$

5 ☐ $\frac{1}{8}$

6 ☐ Do not know

Continue at page 8

Exercise 13

One assumes that the maximum monthly level of water at a dike is given by the cumulative distribution function $F(x) = 1 - \exp\left(-x^{\frac{1}{3}}\right)$ As a rough approximation, we further assume that this distribution is the same every month of the year.

Question 13

The probability of the water level exceeding 8 at least once during a year is found to be

1 ☐ $1 - \Phi\left(\frac{8-6\sqrt{8}}{12\sqrt{38}}\right)$

2 ☐ $\left(1 - e^{-2}\right)^{12}$

3 ☐ e^{-24}

4 ☐ $1 - e^{-24}$

5 ☐ $1 - \left(1 - e^{-2}\right)^{12}$

6 ☐ Do not know

Exercise 14

Consider the pair (X, Y) of continuous random variables with joint density given by $f(x, y) = 4xy$, $0 < x < 1, 0 < y < 1$.

Question 14

The density $f_Z(z)$ of $Z = Y/X$ is

1 ☐ $f_Z(z) = \begin{cases} z & \text{for } 0 < z \leq 1 \\ \frac{1}{z^3} & \text{for } 1 < z \end{cases}$

2 ☐ $f_Z(z) = \int_0^z 4x^2y \mathrm{d}x, \qquad 0 < z$

3 ☐ $f_Z(z) = \frac{1}{z^2}, \qquad 1 < z$

4 ☐ $f_Z(z) = \begin{cases} \frac{2}{3}z & \text{for } 0 < z \leq 1 \\ \frac{2}{3z^2} & \text{for } 1 < z \end{cases}$

5 ☐ $f_Z(z) = \int_0^1 4x^3z \mathrm{d}x, \qquad 0 < z$

6 ☐ Do not know

Continue at page 9

Exercise 15

We have the three discrete random variables X_1, X_2, X_3 where (X_1, X_2, X_3) follows a multinomial distribution with $n = 3$ trials. The probability parameters p_1, p_2, p_3 for X_1, X_2, X_3 are given as $p_1 = 1/2, p_2 = 1/3, p_3 = 1/6$.

Question 15

We find $\mathsf{P}(X_1 + X_3 \leq 2)$ to be

1 ☐ $\sum_{i=0}^{2} \binom{3}{i} \left(\frac{2}{3}\right)^i \frac{1}{3^{3-i}}$

2 ☐ $\sum_{i=0}^{2} \frac{3!}{i!(2-i)!1!} \left(\frac{1}{2}\right)^i \left(\frac{1}{3}\right)^{2-i} \frac{1}{6}$

3 ☐ $\sum_{i=0}^{1} \sum_{j=0}^{1} \frac{3!}{i!j!(3-i-j)!} \left(\frac{1}{2}\right)^i \left(\frac{1}{3}\right)^{2-i-j} \left(\frac{1}{6}\right)^j$

4 ☐ $\sum_{i=0}^{2} \sum_{j=0}^{2} \frac{3!}{i!j!(3-i-j)!} \left(\frac{1}{2}\right)^i \left(\frac{1}{3}\right)^{2-i-j} \left(\frac{1}{6}\right)^j$

5 ☐ $\Phi(0)$

6 ☐ Do not know

Exercise 16

The conditional distribution of the random variable Y given $X = x$ is given by the density $f_Y(y|X = x) = 1/xe^{-y/x}$. That is, for $X = x$, Y is $\exp(1/x)$ distributed. X is gamma(2,1) distributed, i.e. the density $f_X(x)$ of X is $f_X(x) = xe^{-x}$.

Question 16

The mean of Y is

1 ☐ 1

2 ☐ 2

3 ☐ 3

4 ☐ 4

5 ☐ $\frac{3}{2}$

6 ☐ Do not know

Continue at page 10

Exercise 17

The waiting time for calls to an emergency hotline may be described by a density $f(x) = 2xe^{-x^2}$. Five calls are made to the hotline.

Question 17

The third longest waiting time has density

1 ☐ $2xe^{-x^2}$

2 ☐ $x^5e^{-x^2}$

3 ☐ $\binom{5}{3}\left(1-e^{-x^2}\right)^3\left(e^{-x^2}\right)^2$

4 ☐ $120xe^{-x^2}\left(1+\left(e^{-x^2}\right)^2-2e^{-x^2}\right)\left(e^{-x^2}\right)^2$

5 ☐ $60xe^{-3x^2}\left(1-e^{-x^2}\right)^2$

6 ☐ Do not know

Exercise 18

In this problem we consider two independent events.

Question 18

That two events are independent means that

1 ☐ The events may not occur simultaneously

2 ☐ The probability of the union of the events may be determined as the product of the probabilities of each of the two events occurring individually.

3 ☐ The probability of the intersection of the events may be determined as the product of the probabilities of each of the two events occurring individually.

4 ☐ The probability of the union of the events may be determined as the sum of the probabilities of the events.

5 ☐ The indicator function of the union of the events may be determined as the sum of the indicator functions for the two events.

6 ☐ Do not know

Continue at page 11

Exercise 19

The change in the price of a financial security during a single day may be described by a distribution with mean 0 and variance 49.

Question 19

An upper bound for the probability of the price changing by more than 21 in a single day is

1 ☐ $2\Phi(-3)$

2 ☐ $1-\Phi(3)$

3 ☐ $\frac{1}{9}$

4 ☐ $\frac{1}{7}$

5 ☐ $\frac{1}{3}$

6 ☐ Do not know

Exercise 20

The normed dividend from two securities may be described by a standardised bivariate normal distribution with correlation coefficient $\rho = -\frac{1}{2}$

Question 20

The probability that the return from both securities is negative is found to be

1 ☐ $\frac{\frac{\pi}{2}+\arctan(\sqrt{3})}{2\pi}$

2 ☐ $\frac{\frac{\pi}{2}-\arctan(\sqrt{3})}{2\pi}$

3 ☐ $\frac{1}{6}$

4 ☐ $\frac{1}{4}$

5 ☐ $\frac{1}{3}$

6 ☐ Do not know

Continue at page 12

Exercise 21

The pair (X, Y) of continuous random variables has the joint density $f(x, y) = 3e^{-(x+y)}$, for $\frac{1}{2}x \le y \le 2x$.

Question 21

We calculate $\mathsf{E}(\sqrt{XY})$ as

1 ☐ $\mathsf{E}(\sqrt{XY}) = \int_0^\infty \int_0^\infty 3\sqrt{xy}e^{-(x+y)}\mathrm{d}y\mathrm{d}x$

2 ☐ $\mathsf{E}(\sqrt{XY}) = \int_{-\infty}^\infty \int_{-\infty}^\infty 3\sqrt{xy}e^{-(x+y)}\mathrm{d}y\mathrm{d}x$

3 ☐ $\mathsf{E}(\sqrt{XY}) = \int_0^\infty \int_{\frac{1}{2}x}^{2x} 3\sqrt{xy}e^{-(x+y)}\mathrm{d}y\mathrm{d}x$

4 ☐ $\mathsf{E}(\sqrt{XY}) = \int_0^\infty \int_0^\infty 3xy\sqrt{xy}e^{-(x+y)}\mathrm{d}y\mathrm{d}x$

5 ☐ $\mathsf{E}(\sqrt{XY}) = \int_0^\infty \int_{\frac{1}{2}x}^{2x} 3xy\sqrt{xy}e^{-(x+y)}\mathrm{d}y\mathrm{d}x$

6 ☐ Do not know

Exercise 22

A researcher believes he has shown that the lifespan of an oak in years may be described by a distribution with density $f(t) = 3/(1+2t)^{5/2}$. In the following one may assume the researcher is right.

Question 22

The probability of a 100 year old oak tree dying within a small time span, Δt, is found, possibly approximately, as

1 ☐ $\frac{3}{201}\Delta t$

2 ☐ $\frac{\sqrt{201}}{2706867}\Delta t$

3 ☐ $\frac{\sqrt{201}}{40401}$

4 ☐ $1 - (201 + 2\Delta t)^{-\frac{3}{2}}$

5 ☐ $(201 + 2\Delta t)^{-\frac{3}{2}}$

6 ☐ Do not know

Continue at page 13

Exercise 23

We are told that the two random variables X and Y have the properties $\mathsf{E}(X) = 2, \mathsf{V}(X) = 4, \mathsf{E}(Y) = 3, \mathsf{E}(XY) = 6$

Question 23

From this we may deduce the following about X and Y

1 ☐ X and Y are independent

2 ☐ X and Y are uncorrelated

3 ☐ X and Y are positively correlated

4 ☐ X and Y are negatively correlated

5 ☐ None of the above may be determined from the information provided

6 ☐ Do not know

Exercise 24

A sample space Ω is partitioned into three pairwise disjoint events C_1, C_2, C_3, such that their union is the sure event, Ω. We are given that $\mathsf{P}(C_1) = \mathsf{P}(C_2) = \frac{1}{4}$. Further, we consider the event D, which satisfies $\mathsf{P}(D|C_1) = \mathsf{P}(D|C_2) = 1/5, \mathsf{P}(D|C_3) = 2/5$.

Question 24

If the event D has occurred the probability of the event C_1 also having occurred is found to be

1 ☐ $\frac{1}{7}$

2 ☐ $\frac{1}{6}$

3 ☐ $\frac{2}{11}$

4 ☐ $\frac{1}{4}$

5 ☐ The probability cannot be calculated due to insufficient information

6 ☐ Do not know

Continue at page 14

Exercise 25

We assume that bacterial colonies in a given growth medium, which we observe for a given time, are independent of each other with a mean prevalence of 2 colonies per ml. We may assume that the colonies have vanishing size

Question 25

The probability of not finding a bacterial colony in $\frac{1}{4}$ ml of the growth medium after the given period of time is

1 ☐ 0,607

2 ☐ 0,500

3 ☐ 0,368

4 ☐ 0,250

5 ☐ 0,125

6 ☐ Do not know

Exercise 26

A discrete random variable N is uniformly distributed on $1, 2, \ldots, n$ while for given $N = i$ the random random variable X follows a binomial distribution with counting parameter $n = i$ and probability parameter $\frac{1}{i}$.

Question 26

We determine $\mathsf{P}(X = n)$ as

1 ☐ $\sum_{i=1}^{n} \frac{1}{n} \binom{n}{i} \frac{1}{n}^{i} \left(1 - \frac{1}{n}\right)^{n-i}$

2 ☐ $\binom{n+1}{2} \frac{1}{n^n}$

3 ☐ $\frac{1}{n^n}$

4 ☐ $\frac{1}{n^{n+1}}$

5 ☐ $(n-1)\frac{1}{n^n}$

6 ☐ Do not know

Continue at page 15

Exercise 27

The function $f(x) = c(x^2 - 4x + 3)e^{-x}$ is proposed as a density for a continuous random variable.

Question 27

We may say the following about the function $f(x)$

1 ☐ $f(x)$ may be a density for all positive x and a positively chosen c

2 ☐ $f(x)$ may be a density for all positive x and a negatively chosen c

3 ☐ $f(x)$ may be a density for all negative x and a positively chosen c

4 ☐ $f(x)$ may be a density for all negative x and a negatively chosen c

5 ☐ $f(x)$ cannot be a density on the entirety of $\mathbb{R}_-$ or $\mathbb{R}_+$.

6 ☐ Do not know

Exercise 28

The random variable X is geometrically distributed $\mathsf{P}(X = x) = \frac{1}{5}\left(\frac{4}{5}\right)^{x-1}$, while Y independently of X follows a Poisson distribution $\mathsf{P}(Y = y) = \frac{2^y}{y!}e^{-2}$.

Question 28

We find $\mathsf{E}(XY) + \mathsf{E}(X)\mathsf{E}(Y)$ to be

1 ☐ 0

2 ☐ 10

3 ☐ 16

4 ☐ 20

5 ☐ 32

6 ☐ Do not know

Continue at page 16

Exercise 29

A clinic at a hospital calls in patients for treatment. For every patient they summon there is a probability the patient will not come. We wish to determine the number of patients to be summoned if the probability of more than 10 patients coming is to be more than 90%.

Question 29

The distribution which is best suited to perform the calculation stated above is

1 ☐ The normal distribution

2 ☐ The gamma distribution

3 ☐ The Poisson distribution

4 ☐ The binomial distribution

5 ☐ The negative binomial distribution

6 ☐ Do not know

Exercise 30

Among many other things, Australia exports iron ore and coal. The amount of iron ore and coal that is exported a given year can be described by a bivariate normal distribution with the following parameters $\mathsf{E}(X) = 500, \mathsf{E}(Y) = 200, \mathsf{V}(X) = 2500, \mathsf{V}(Y) = 1600$ and correlation coefficient $\rho = \frac{4}{5}$. We note that the units above are in millions of tons, and that X describes the coal and Y the iron ore.

Question 30

What is the expected amount of iron ore that will be exported in a year where 160 million tons of coal are exported.

1 ☐ 540

2 ☐ 500

3 ☐ 475

4 ☐ 460

5 ☐ 450

6 ☐ Do not know

End of exam