Written examination: May 24 2012 *Course no.* : 02405

Name of course: Probability theory *Duration* : 4 hours

Aids allowed: All

The questions have been answered by:

________________________ ________________________ ________

(name) (signature) (table no.)

There are 30 exercises with a total of 30 questions. The numbering of the exercises are given as 1,2,. . . , 30 in the text. Every single question is also numbered and given as question 1,2,. . . ,30 in the text. Please give your answers to the 30 questions in the table below.

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
answer															

Question	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
answer															

Each question is a multiple choice question, where the options are numbered from 1 to 6. Only one number should be given for each question. If more than one number is given, the question is considered to be "not answered". If, by mistake, a wrong number has been given, it can be corrected by "blackening" out the wrong answer and writing the correct number below. If there is any doubt about the meaning of a correction, the question will be considered to be "not answered". **Only the front page is to be handed in**. In case the questionnaire is blank or if the examination is left before it officially ends, this first page has to be returned to the officials before leaving the examination room. Draft and intermediate calculations will not be taken into account. Only the numbers written in the table above will be scored.
5 points are obtained for a correct number, and -1 point is given for a wrong number. If a question is left unanswered or if the number is "6" (for 'do not know') 0 points are given. The number of points, corresponding to the specific marks, needed to pass will be determined in connection with the examination of the papers.
Remember to write your name, your signature and the number of your table at the questionnaire before handing it in.
Notice, the idea behind the exercises is that there is one and only one correct number for every single question. All the given number options may not necessarily make sense. The last page of the exam set is page nr 16; flip to that page to be sure it is there.
Please note, that that the notation $\log(\cdot)$ is used for the natural logarithm, i.e. logarithms with base e, while Φ denotes the cumulative distribution function for a standardised normally distributed random variable.

Exercise 1

If a specific type of nut is infected by a certain species of fungus and a staph-bacteria, a certain powerful carcinogenic might be produced. The fungus occurs with a frequency of $\frac{1}{400}$, while the bacteria occurs with frequency $\frac{1}{900}$. The probability of a nut with the fungus also having the bacteria is $\frac{1}{3}$.

Question 1

The probability of a nut having both bacterial and fungal infections is found to be

1 ☐ $\frac{1}{1200}$

2 ☐ $\frac{1}{36000}$

3 ☐ $\frac{13}{3600}$

4 ☐ $\frac{1}{3}$

5 ☐ $\frac{1}{2700}$

6 ☐ Do not know

Exercise 2

We look at the time it takes from a control message is sent from a control station, until it has been correctly received and processed by a satellite. Due to transmission errors, this time is highly variable with a mean of 1s, and a variance of 16s^2.

Question 2

A best upper bound for the probability of the message not being processed correctly within 25s is found to be

1 ☐ $\frac{1}{6}$

2 ☐ $\frac{1}{36}$

3 ☐ $\Phi(-6)$

4 ☐ $\frac{1}{25}$

5 ☐ $\Phi(-4)$

6 ☐ Do not know

Exercise 3

A certain type of ball bearings is produced in batches of 400. The individual ball-bearing passes quality inspection with a probability of $\frac{4}{5}$ independently of the other bearings.

Question 3

The standard deviation of the amount of ball-bearings passing inspection in a batch is determined as

1 ☐ 320

2 ☐ 64

3 ☐ 32

4 ☐ 16

5 ☐ 8

6 ☐ Do not know

Exercise 4

In laser surgery of the vocal chords, it is attempted to calibrate the laser beam. Assuming the calibration point as the origin of a coordinate system, the true location of the laser beam can be described by the coordinates (X, Y), which can be considered as independent standardised, normally distributed, random variables.

Question 4

The probability of the laser beam being more than 2 units of length from the calibration point, is found to be

1 ☐ $1 - e^{-2}$

2 ☐ $\frac{1}{4}$

3 ☐ $\Phi(-\sqrt{2})^2$

4 ☐ $\Phi(-2)^2$

5 ☐ e^{-2}

6 ☐ Do not know

Exercise 5

For two standardised random variables X and Y we have $\mathrm{E}(XY) = \frac{1}{2}$.

Question 5

$\mathrm{Var}(X - Y)$ is

1 ☐ 1

2 ☐ 2

3 ☐ 3

4 ☐ $\frac{1}{2}$

5 ☐ The problem cannot be solved, due to missing information.

6 ☐ Do not know

Exercise 6

In an outcome space Ω the events $B_i,\ i = 1, 2, 3$ satisfy the conditions $B_i \cap B_j = \emptyset$ for $i \neq j$ and $B_1 \cup B_2 \cup B_3 = \Omega$. Further, we have $P(B_1) = \frac{1}{2}$, $P(B_2) = \frac{1}{3}$, and $P(B_3) = \frac{1}{6}$. Finally we have, for the event A, that $P(A|B_i) = \frac{1}{i}$.

Question 6

We find $P(B_2|A)$ to be

1 ☐ $\frac{1}{2}$

2 ☐ $\frac{13}{18}$

3 ☐ $\frac{3}{13}$

4 ☐ $\frac{1}{6}$

5 ☐ The problem cannot be solved, since $P(A|B_i)$ isn't a probability distribution over $i = 1, 2, 3$.

6 ☐ Do not know

Exercise 7

The probability of a flight being delayed at an airport is 0.2. As an approximation, the individual departures can be seen as being independent of each other.

Question 7

Find, possibly approximately, the probability that at most 50 out of 200 departures are delayed.

1 ☐ $\sum_{i=0}^{50} \binom{50}{i} 0.2^i 0.8^{50-i}$

2 ☐ $\sum_{i=0}^{50} \frac{160^i}{i!} e^{-160}$

3 ☐ $\Phi(1.74)$

4 ☐ $\Phi(1.62)$

5 ☐ $\Phi(1.86)$

6 ☐ Do not know

Exercise 8

We have a random variable X, which is uniformly distributed on the interval $]0;3[$. We introduce the random variable $Y = -\log\left(\frac{X}{3}\right)$.

Question 8

The density $f_Y(y)$ of Y is found to be

1 ☐ $e^{-y}, \quad 0 < y$

2 ☐ $3e^{-3y}, \quad 0 < y$

3 ☐ $\log(3), \quad y \in]0, \frac{1}{\log(3)}[$

4 ☐ $e^{y}, \quad y < 0$

5 ☐ $3e^{3y}, \quad y < 0$

6 ☐ Do not know

Exercise 9

Of the random variables X and Y, it is known that $E(Y|X = x) = e^x$, and that the marginal density $f_X(x)$ of X is $f_X(x) = 2e^{-2x}$.

Question 9

We find $E(Y)$ to be

1 ☐ 1

2 ☐ 2

3 ☐ e^2

4 ☐ $e^{\frac{1}{2}}$

5 ☐ ∞

6 ☐ Do not know

Exercise 10

A fence has been erected to keep mink out of a breeding ground for birds. The fence can keep mink with a size greater than 5 out. The distribution of the size of mink can be described by a normal(9,4) distribution.

Question 10

The probability of at least one out of ten random mink being able to go through the fence is found as

1 ☐ 0.023

2 ☐ 0.206

3 ☐ 0.228

4 ☐ 0.264

5 ☐ 0.284

6 ☐ Do not know

Exercise 11

A drone, a male bee, is required for an experiment. It is known that one out of twenty bees of this species are drones.

Question 11

The probability of having to investigate at least ten bees before finding a drone is found as

1 ☐ $\binom{20}{10}\left(\frac{19}{20}\right)^9\left(\frac{1}{20}\right)^{11}$

2 ☐ $\frac{1}{20}\left(\frac{19}{20}\right)^9$

3 ☐ $\left(\frac{19}{20}\right)^9$

4 ☐ $\sum_{i=10}^{\infty}\binom{20}{10}\left(\frac{19}{20}\right)^{10}\left(\frac{1}{20}\right)^{i-10}$

5 ☐ $\frac{1}{2}$

6 ☐ Do not know

Exercise 12

For patients with a specific type of heart disease, we know that the risk of a patient having elevated blood sugar is $\frac{3}{5}$, the probability of a patient having elevated blood pressure and elevated blood sugar is $\frac{2}{5}$ while the probability of a patient having at least one of the symptoms elevated blood sugar and elevated blood pressure is $\frac{4}{5}$.

Question 12

The probability of a patient with the heart disease having elevated blood pressure is

1 ☐ $\frac{1}{5}$

2 ☐ $\frac{2}{5}$

3 ☐ $\frac{3}{5}$

4 ☐ $\frac{2}{3}$

5 ☐ $\frac{4}{5}$

6 ☐ Do not know

Exercise 13

A public department has requested five citizens to arrive between 9 and 10 o'clock.

Question 13

The probability of at least four citizens having arrived by 9:30 is found to be

1 ☐ $\frac{1}{16}$

2 ☐ $\frac{3}{32}$

3 ☐ $\frac{5}{32}$

4 ☐ $\frac{3}{16}$

5 ☐ $\frac{1}{2}$

6 ☐ Do not know

Exercise 14

A bulk carrier is laden with coal from two different places. Mean and variance of the cargo loaded at both places are respectively $10,000$ tonnes and $(5,000 \text{ tonnes})^2$, with correlation coefficient $\frac{3}{5}$. It can be assumed that the joint distribution is a bivariate normal distribution.

Question 14

The probability of the total cargo from the two locations exceeding 30,000 tonnes is found to be

1 ☐ $1 - \Phi(2)$

2 ☐ $\frac{\text{Arctan}\left(\sqrt{\frac{3}{4}}\right)}{2\pi}$

3 ☐ $\Phi\left(-\sqrt{\frac{7}{3}}\right)$

4 ☐ $\frac{\text{Arctan}\left(\sqrt{\frac{5}{3}}\right)}{2\pi}$

5 ☐ $1 - \Phi\left(\frac{\sqrt{5}}{2}\right)$

6 ☐ Do not know

Exercise 15

Of a certain type of electronics, for which the lifetime is denoted by T, it is known that

$$\lim_{\Delta t \to 0} \frac{P(t < T < t + \Delta t || T > t)}{\Delta t} = e^{-t}.$$

Question 15

We find $P(T > t)$ as

1 ☐ e^{-t}

2 ☐ $\int_t^\infty e^{-u} \mathrm{d}u$

3 ☐ $(1+t)e^{-t}$

4 ☐ $\exp\left(-\int_0^t e^{-u} \mathrm{d}u\right)$

5 ☐ $\int_0^t e^{-u} \mathrm{d}u$

6 ☐ Do not know

Exercise 16

At a large military hospital, there are ten orthopaedic surgeons on call, four of whom are knee specialists. A soldier is brought to the hospital for an acute operation, and two surgeons are randomly selected among the ten.

Question 16

The probability of both selected surgeons being knee specialists is best found using the

1 ☐ Poisson distribution

2 ☐ Negative binomial distribution

3 ☐ Geometric distribution

4 ☐ Hypergeometric distribution

5 ☐ Binomial distribution

6 ☐ Do not know

Exercise 17

Consider a circle centred at the origin, (0,0). A point is chosen randomly within the part of the disc given by the circle that lies in the first quadrant. The coordinates of the point are denoted by (X, Y).

Question 17

We find $P\left(X \tan\left(\frac{\pi}{6}\right) < Y < X \tan\left(\frac{\pi}{3}\right)\right)$ as

1 ☐ $\frac{1}{9}$

2 ☐ $2\frac{\text{Arctan}(\frac{3}{2})}{\pi}$

3 ☐ $2\frac{\text{Arctan}(\frac{1}{2})}{\pi}$

4 ☐ $\frac{1}{4}$

5 ☐ $\frac{1}{3}$

6 ☐ Do not know

Exercise 18

The discrete random variable X follows a geometric distribution with parameter p, ie.
$P(X = x) = p(1-p)^{x-1}$. The random variable Y is binomially distributed for a given $X = x$, with probability parameter q and number of trials equal to x.

Question 18

We determine $P(Y = 0)$ to be

1 ☐ $(1-p)(1-q)$

2 ☐ $\frac{(1-q)p}{1-(1-q)(1-p)}$

3 ☐ $\frac{q(1-p)}{1-(1-q)(1-p)}$

4 ☐ $e^{-(1-p)(1-q)}$

5 ☐ $\frac{(1-q)(1-p)}{1-(1-q)(1-p)}$

6 ☐ Do not know

Exercise 19

A small ferry leaves the ferry landing at most fifteen minutes after having arrived. If four cars have arrived, which corresponds to the maximum capacity of the ferry, the ferry leaves just as the fourth car has boarded. On average the cars arrive at the landing every 3rd minute, independently of each other.

Question 19

The probability of the ferry leaving 15 minutes after having arrived is

1 ☐ $\frac{4}{5}$

2 ☐ 0.22

3 ☐ $\frac{118}{3}e^{-5}$

4 ☐ $\sum_{i=0}^{3}\binom{5}{i}\left(\frac{3}{5}\right)^{i}\left(\frac{3}{5}\right)^{5-i}$

5 ☐ $1-\Phi\left(\frac{5}{3}\right)$

6 ☐ Do not know

Exercise 20

The length of the legs of a cheetah, where the fore legs are assumed to be of equal length and the hind legs are also assumed to be of equal length, can be described by a bivariate normal distribution. For a particular sub-species the forelegs have a mean length of 60cm, and the hind legs have a mean length of 55cm, both with a standard deviation of 1.5cm. The correlation coefficient between the length of the fore and hind legs is known to be 0.85.

Question 20

If a cheetah of this subspecies is known to have forelegs with a length of 63cm, what is the expected length of a hind leg

1 ☐ 58cm

2 ☐ 55cm

3 ☐ 57.1675cm

4 ☐ 57.625cm

5 ☐ 57.55cm

6 ☐ Do not know

Exercise 21

On a hot summer day a kiosk owner sells mainly ice creams and magazines. The number of ice creams sold can be described by a Poisson(12) distribution, and the number of magazines sold can be described by a binomial$\left(30, \frac{1}{4}\right)$ distribution. The ice creams are sold for 15 kr. each, and the magazines for 35 kr. each.

Question 21

The expected revenue from sales of ice cream and magazines on a summer day is

1 ☐ 442.50 kr.

2 ☐ 487.50 kr.

3 ☐ 502.50 kr.

4 ☐ 532.50 kr.

5 ☐ 557.50 kr.

6 ☐ Do not know

Exercise 22

When spray-painting a ship two types of problems occur. One of these is the formation of small air bubbles underneath the paint, which leaves the paint at this location prone to falling off and leave the steel exposed. Another is the formation of small grains that dry slowly, which also have a tendency to scale off later and expose the steel. The air bubbles occur with a frequency of $\frac{1}{10}$ pr. m^2, and the grains occur with a frequency of $\frac{1}{5}$ pr. m^2 independently of each other.

Question 22

The probability of neither error occurring on a 10 m^2 area of a ship's hull is

1 ☐ $6e^{-3}$

2 ☐ $\left(\frac{9}{10} \cdot \frac{19}{20}\right)^{10}$

3 ☐ e^{-2}

4 ☐ e^{-3}

5 ☐ $1 - \left(\frac{1}{10} \cdot \frac{1}{20}\right)^{10}$

6 ☐ Do not know

Exercise 23

A normal six-sided die is rolled twice. Let the random variable X denote the lowest number on the top face in the two throws, and the random variable Y the largest number on the top face in the throws.

Question 23

We find $P(X + Y = 7)$ to be

1 ☐ $\frac{1}{7}$

2 ☐ $\frac{73}{648}$

3 ☐ $\frac{1}{6}$

4 ☐ $\frac{5}{36}$

5 ☐ $\frac{7}{36}$

6 ☐ Do not know

Exercise 24

The random variables X and Y follow a bivariate distribution with joint probability density

$$f(x,y) = \begin{cases} 3e^{-(x+y)} & \text{for } 0 < \frac{x}{2} < y < 2x \\ 0 & \text{else} \end{cases}.$$

Question 24

We find the probability $P\left(X \le \frac{1}{2}, Y \le 1\right)$ to be

1 ☐ $3\left(1 - e^{-2} - e^{-\frac{1}{2}} + e^{-\frac{5}{2}}\right)$

2 ☐ $\left(1 - e^{-\frac{3}{4}}\right)^2$

3 ☐ $2 - 3e^{-\frac{1}{2}} + e^{-\frac{3}{2}}$

4 ☐ $2\left(1 - e^{-\frac{3}{4}}\right) + 3\left(e^{-\frac{5}{2}} - e^{-2}\right)$

5 ☐ $3e^{-\frac{3}{2}}$

6 ☐ Do not know

Exercise 25

A random variable X follows a beta(2,2) distribution with density $f_X(x) = 6x(1-x)$. Further, it is known that for a discrete random variable Y, $P(Y = y|X = x) = \binom{2}{y} x^y (1-x)^{2-y}$.

Question 25

We determine $P(Y = 2)$ as

1 ☐ $\frac{2}{5}$

2 ☐ $\frac{1}{4}$

3 ☐ $\frac{3}{10}$

4 ☐ $\frac{5}{12}$

5 ☐ $\frac{3}{4}$

6 ☐ Do not know

Exercise 26

The Danish National Lottery is contemplating introducing a new game, where five numbers are drawn in the range 1-50. All the numbers between 1 and 50 are present with the same probability in each draw, no matter the result of the previous draws. There is no reward if all five numbers are different in all other cases some reward is obtained.

Question 26

The probability of gaining a reward in the game is

1 ☐ 18.6%

2 ☐ 7.8%

3 ☐ 9.6%

4 ☐ 20.2%

5 ☐ 23.4%

6 ☐ Do not know

Exercise 27

The pair of random variables (X, Y) follows a standardised bivariate normal distribution with correlation coefficient $-\frac{5}{13}$.

Question 27

The probability $P(X < Y < 0)$ is

1 ☐ $\frac{\text{Arctan}(\frac{3}{2})}{2\pi}$

2 ☐ $\frac{\text{Arctan}\left(\frac{\sqrt{3}}{3}\right)}{\pi}$

3 ☐ $\frac{\text{Arctan}(\frac{2}{3})}{2\pi}$

4 ☐ $\frac{\text{Arctan}\left(\frac{\sqrt{3}}{2}\right)}{2\pi}$

5 ☐ $\frac{1}{3}$

6 ☐ Do not know

Exercise 28

The field strength of an electromagnetic signal can, on a suitably normalised scale, be described by a Rayleigh distributed random variable.

Question 28

Determine, possibly approximately, the probability of the field strength lying between 2 and 2.01.

1 ☐ $2e^{-2}$

2 ☐ e^{-2}

3 ☐ $\frac{e^{-2}}{100}$

4 ☐ $\frac{e^{-2}}{50}$

5 ☐ $e^{-1} - e^{-2}$

6 ☐ Do not know

Exercise 29

One wants to determine, if two continuous random variables are independent.

Question 29

Independence may be determined by means of the knowledge of

1 ☐ the marginal densities of the two variables

2 ☐ the joint probability density of the two variables

3 ☐ the conditional expectations of the two variables

4 ☐ the correlation coefficient between the two variables.

5 ☐ the united event of the two variables.

6 ☐ Do not know

Exercise 30

The random variables X and Y follow a bivariate distribution with joint density

$$f(x,y) = \begin{cases} 3e^{-(x+y)} & \text{for } 0 < \frac{x}{2} < y < 2x \\ 0 & \text{ellers} \end{cases} .$$

We introduce the random variable $Z = \frac{Y}{X}$ with density $f_Z(z)$.

Question 30

The density $f_Z(z)$ is

1 ☐ $f_Z(z) = \frac{3}{(1+z)^2}, \quad 0 < z$

2 ☐ $f_Z(z) = \frac{3}{(1+z)^2}, \quad \frac{1}{2} < z < 2$

3 ☐ $f_Z(z) = \int_{\frac{z}{3}}^{\frac{2z}{3}} xe^{-(x+xz)} \mathrm{d}z, \quad 0 < z$

4 ☐ $f_Z(z) = \int_{\frac{z}{3}}^{\frac{2z}{3}} xe^{-(x+xz)} \mathrm{d}z, \quad \frac{1}{2} < z < 2$

5 ☐ $f_Z(z) = \frac{1}{(1+z)^2}, \quad 0 < z$

6 ☐ Do not know

End of the exam.