

**Natural Deduction
for Classical Propositional Logic**

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Natural Deduction

- ND: System for structured deduction from a set of assumptions, based on rules, specific to the logical connectives.
- For each logical connective: **introduction rule(s)** and **elimination rule(s)**
- **Introduction** (opening) and **cancelation** (closing, discharge) of **assumptions**.
Assumptions can be re-used many times before canceled.
- Cancelation of assumptions: only when the rules allow it, but not an obligation.
- All open assumptions at the end of the derivation must be declared.
The less assumptions, the stronger the derivation.

ND rules for the propositional connectives

Introduction rules:

$$(\wedge I) \quad \frac{A, B}{A \wedge B}$$

$$(\vee I) \quad \frac{A}{A \vee B}, \frac{B}{A \vee B}$$

Elimination rules:

$$(\wedge E) \quad \frac{A \wedge B}{A}, \frac{A \wedge B}{B}$$

$$(\vee E) \quad \frac{A \vee B \quad \begin{array}{c} [A] \quad [B] \\ \vdots \quad \vdots \\ C \quad C \end{array}}{C}$$

Introduction rules:

$$(\rightarrow I) \frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \rightarrow B}$$

$$(\neg I) \frac{\begin{array}{c} [A] \\ \vdots \\ \perp \end{array}}{\neg A}$$

Elimination rules:

$$(\rightarrow E) \frac{A, A \rightarrow B}{B}$$

$$(\neg E) \frac{A, \neg A}{\perp}$$

Two more ND rules

Ex falsum quodlibet:

$$(\perp) \quad \frac{\perp}{A}$$

Reductio ad absurdum:

$$(RA) \quad \frac{\begin{array}{c} [\neg A] \\ \vdots \\ \perp \end{array}}{A}$$

Propositional Natural Deduction: Example 1

$A \wedge B \vdash_{\text{ND}} B \wedge A :$

$$\begin{array}{c}
 (\wedge E) \frac{A \wedge B}{B} \quad (\wedge E) \frac{A \wedge B}{A} \\
 (\wedge I) \frac{\quad}{B \wedge A}
 \end{array}$$

Propositional Natural Deduction: Examples 2

$\vdash_{\text{ND}} A \rightarrow \neg\neg A :$

$$\begin{array}{c}
 (\neg E) \frac{[A]^2, [\neg A]^1}{\perp} \\
 (\neg I) \frac{\perp}{\neg\neg A} \quad 1 \\
 (\rightarrow I) \frac{\neg\neg A}{A \rightarrow \neg\neg A} \quad 2
 \end{array}$$

$\vdash_{\text{ND}} \neg\neg A \rightarrow A :$

$$\begin{array}{c}
 (\neg E) \frac{[\neg\neg A]^2, [\neg A]^1}{\perp} \\
 (RA) \frac{\perp}{A} \quad 1 \\
 (\rightarrow I) \frac{A}{\neg\neg A \rightarrow A} \quad 2
 \end{array}$$

Propositional Natural Deduction: Examples 3

$A \rightarrow B \vdash_{\text{ND}} \neg B \rightarrow \neg A :$

$$\begin{array}{c}
 \frac{[A]^1, A \rightarrow B}{B}, [\neg B]^2 \\
 \hline
 \perp \\
 \hline
 \neg A \quad 1 \\
 \hline
 \neg B \rightarrow \neg A \quad 2
 \end{array}$$

$\neg B \rightarrow \neg A \vdash_{\text{ND}} A \rightarrow B :$

$$\begin{array}{c}
 \frac{[\neg B]^1, \neg B \rightarrow \neg A}{[\neg A]}, [A]^2 \\
 \hline
 \frac{\perp}{B} \text{---} 1 \\
 (\rightarrow I) \frac{B}{A \rightarrow B} \text{---} 2
 \end{array}$$

Propositional Natural Deduction: Example 4

$A \vee B \vdash_{\text{ND}} \neg A \rightarrow B :$

$$\begin{array}{c}
 \frac{[\neg A]^1, [A]^3}{\perp} \\
 \\
 \frac{A \vee B \quad \frac{B}{\neg A \rightarrow B} 1 \quad \frac{[\neg A]^2, [B]^3}{\neg A \rightarrow B} 2}{\neg A \rightarrow B} 3
 \end{array}$$

Propositional Natural Deduction: Example 5

$\vdash_{\text{ND}} (A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C) :$

$$\begin{array}{c}
 \begin{array}{c}
 (\wedge E) \frac{[A \wedge B]^1}{B} \quad (\rightarrow E) \frac{(\wedge E) \frac{[A \wedge B]^1}{A}, [A \rightarrow (B \rightarrow C)]^2}{B \rightarrow C}}{C} \\
 (\rightarrow I) \frac{C}{(A \wedge B) \rightarrow C} \quad 1 \\
 (\rightarrow I) \frac{(\rightarrow I) \frac{C}{(A \wedge B) \rightarrow C} \quad 1}{(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C)} \quad 2
 \end{array}
 \end{array}$$

Propositional Deduction: a challenge!

Derive in ND the formula:

$$p \vee \neg p$$

Award for the first correct entry!