First-order logic:
Free and bound variables. Scope of a quantifier. Substitution of terms for variables. Capture. Variable renaming.

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Free and bound variables

Two essentially different ways in which we use individual variables in first-order formulae:

1. Free variables: used to denote *unknown or unspecified objects*, as in $(\mathbf{5} < x) \vee (x^2 + x - \mathbf{2} = \mathbf{0})$.

2. Bound variables: used *to quantify*, as in
   $\exists x((\mathbf{5} < x) \vee (x^2 + x - \mathbf{2} = \mathbf{0}))$
   and $\forall x((\mathbf{5} < x) \vee (x^2 + x - \mathbf{2} = \mathbf{0}))$.

Note that the same variable can be *both free and bound in a formula*, e.g. x in the formula $x > \mathbf{0} \wedge \exists x(\mathbf{5} < x)$.

A formula with no bound variables is an open formula.
A formula with no free variables is a closed formula, or a sentence.

Scope of a quantifier

Scope of (an occurrence of a) quantifier in a given formula A: the *unique* subformula QxB beginning with that occurrence of the quantifier. E.g.:

$$\underline{\forall x}((x > \mathbf{5}) \rightarrow \forall y(y < \mathbf{5} \rightarrow (y < x \land \exists x(x < \mathbf{3})))).$$

$$\forall x((x > \mathbf{5}) \rightarrow \underline{\forall y}(y < \mathbf{5} \rightarrow (y < x \land \exists x(x < \mathbf{3})))).$$

$$\forall x((x > \mathbf{5}) \rightarrow \forall y(y < \mathbf{5} \rightarrow (y < x \land \underline{\exists x}(x < \mathbf{3})))).$$

A bound occurrence of a variable x is bound by the *innermost* occurrence of a quantifier Qx in the scope of which it occurs. E.g.:

$$\forall x((x > \mathbf{5}) \rightarrow \forall y(y < \mathbf{5} \rightarrow (y < x \land \exists x(x < \mathbf{3})))),$$

while

$$\forall x((x > \mathbf{5}) \rightarrow \forall y(y < \mathbf{5} \rightarrow (y < x \land \exists x(x < \mathbf{3})))).$$

Using bound and free variables in a formula

Free variables have their own values in a given formula (determined by a variable assignment), while bound variables only play a dummy role and can be replaced (with care!) by one another.

For instance, the sentence

$$\exists x(\mathbf{5} < x \wedge x^2 + x - \mathbf{2} = \mathbf{0})$$

means exactly the same as

$$\exists y(\mathbf{5} < y \wedge y^2 + y - \mathbf{2} = \mathbf{0})$$

Likewise, $\forall x(\mathbf{5} < x \vee x^2 + x - \mathbf{2} = \mathbf{0})$
means the same as $\forall y(\mathbf{5} < y \vee y^2 + y - \mathbf{2} = \mathbf{0})$.

On the other hand, the meaning of

$$\mathbf{5} < x \wedge x^2 + x - \mathbf{2} = \mathbf{0}$$

is essentially different from the meaning of

$$\mathbf{5} < y \wedge y^2 + y - \mathbf{2} = \mathbf{0}.$$

Reusing variables as free and bound in a formula

The same variable can occur both free and bound in a formula:

$$x > \mathbf{5} \rightarrow \forall x(\mathbf{2}x > x).$$

However, the free occurrence of x has nothing to do with the bound occurrences of x:

$$x > \mathbf{5} \rightarrow \forall x(\mathbf{2}x > x).$$

Thus, the formula above has the same meaning as

$$x > \mathbf{5} \rightarrow \forall y(\mathbf{2}y > y),$$

but not the same meaning as

$$y > \mathbf{5} \rightarrow \forall x(\mathbf{2}x > x).$$

Binding a variable by different quantifiers in a formula

Different occurrences of the same variable can be bound by different quantifiers:

$$\exists x(x > \mathbf{5}) \vee \forall x(\mathbf{2}x > x).$$

Again, the occurrences of x, bound by the first quantifier, have nothing to do with those bound by the second one.

For instance, the two x's claimed to exist in the formula

$$\exists x(x > \mathbf{5}) \wedge \exists x(x < \mathbf{3}).$$

need not (and, in fact, cannot) be the same.

Thus, the formula above has the same meaning as each of

$$\exists y(y > \mathbf{5}) \wedge \exists x(x < \mathbf{3}),$$

$$\exists x(x > \mathbf{5}) \wedge \exists z(z < \mathbf{3}),$$

$$\exists y(y > \mathbf{5}) \wedge \exists z(z < \mathbf{3}).$$

Nested bindings of a variable in a formula

Different bindings of the same variable can be nested, e.g.:

$$\forall x(x > \mathbf{5} \rightarrow \exists x(x < \mathbf{3})).$$

Again, the occurrences of x in the subformula $\exists x(x < \mathbf{3})$ are bound by $\exists$ and not related to the first two occurrences of x, bound by $\forall$:

$$\forall x(x > \mathbf{5} \rightarrow \exists x(x < \mathbf{3})).$$

Thus, the formula above has the same meaning as each of

$$\forall x(x > \mathbf{5} \rightarrow \exists y(y < \mathbf{3})),$$

$$\forall z(z > \mathbf{5} \rightarrow \exists x(x < \mathbf{3})),$$

$$\forall z(z > \mathbf{5} \rightarrow \exists y(y < \mathbf{3})).$$

Renaming of a bound variable in a formula

Using the same variable for different purposes in a formula can be confusing, and is often unwanted, so we may want to eliminate it.

Renaming of the variable x in a formula A is the substitution of *all occurrences of* x *bound by the same occurrence of a quantifier* in A with another variable, not occurring in A.

E.g., a possible renaming of $(x > \mathbf{5}) \land \forall x(x > \mathbf{5} \to \neg\exists x(x < y))$
is the formula $(x > \mathbf{5}) \land \forall x(x > \mathbf{5} \to \neg\exists z(z < y))$

However, neither of the following formulae is a correct renaming:

$$(z > \mathbf{5}) \land \forall x((x > \mathbf{5}) \to \neg\exists x(x < y)),$$

$$(x > \mathbf{5}) \land \forall z((z > \mathbf{5}) \to \neg\exists z(z < y)),$$

$$(x > \mathbf{5}) \land \forall x(x > \mathbf{5} \to \neg\exists y(y < y)).$$

PROPOSITION: The result of renaming a variable in a formula is logically equivalent to that formula.

Clean formulae

A formula A is clean if no variable occurs both free and bound in A and every two occurrences of quantifiers bind different variables.

Thus, $\exists x(x > \mathbf{5}) \wedge \exists y(y < z)$ is clean,

while $\exists x(x > \mathbf{5}) \wedge \exists y(y < x)$

and $\exists x(x > \mathbf{5}) \wedge \exists x(y < x)$ are not.

PROPOSITION: Every formula can be transformed into a clean formula by means of several consecutive renamings of variables.

E.g.,

$$(x > \mathbf{5}) \wedge \forall x((x > \mathbf{5}) \rightarrow \neg\exists x(x < y))$$

can be transformed into a clean formula as follows:

$$(x > \mathbf{5}) \wedge \forall x_1((x_1 > \mathbf{5}) \rightarrow \neg\exists x(x < y)),$$

$$(x > \mathbf{5}) \wedge \forall x_1((x_1 > \mathbf{5}) \rightarrow \neg\exists x_2(x_2 < y)).$$

Substitution of a term for a variable in a formula

Unform substitution of a term t for a variable x in a formula A means that all *free* occurrences x in A are simultaneously replaced by t. The result of the substitution is denoted $A[t/x]$.

Example: given the formula

$$A = \forall x(P(x, y) \rightarrow (\neg Q(y) \vee \exists y P(x, y)))$$

we have

$$A[f(y, z)/y] = \forall x(P(x, f(y, z)) \rightarrow (\neg Q(f(y, z)) \vee \exists y P(x, y))),$$

while

$$A[f(y, z)/x] = A$$

because x does not occur free in A.

Intuitively, $A[t/x]$ is supposed to say about the individual denoted by t the same as what A says about the individual denoted by x.

Question: is that always the case?
Is a substitution of a term for a formula always 'safe'?

Capture of a variable in substitution

The formula $A = \exists y(x < y)$ is true in $\mathcal{N}$ *for any value of* x.

However, $A[(y + \mathbf{1})/x] = \exists y(y + \mathbf{1} < y)$, which is false in $\mathcal{N}$.

Therefore, the formula $A[(y + \mathbf{1})/x]$ does not say about the term $y + \mathbf{1}$ the same as what A says about x.

What went wrong?

The occurrence of y in the term $y + \mathbf{1}$ got *captured* by the quantifier $\exists y$, because we mixed the free and the bound uses of y.

Capture: new occurrences of a variable y in the scope of a quantifier Qy introduced as a result of substitution of a term t containing y for another variable x in a formula.

Terms free for substitution for a variable in a formula

A term t is free for (substitution for) a variable x in a formula A, if no variable in t is captured by a quantifier when t is substituted for x in A. Examples:

The term $f(x, y)$ is free for substitution for y in the formula $A = \forall x(P(x, z) \wedge \exists y Q(y)) \rightarrow P(y, z)$,

resulting in $A[f(x, y)/y] = \forall x(P(x, z) \wedge \exists y Q(y)) \rightarrow P(f(x, y), z)$,

but it is not free for substitution for z in A, resulting in

$$A[f(x, y)/z] = \forall x(P(x, f(x, y)) \wedge \exists y Q(y)) \rightarrow P(y, f(x, y)),$$

because a capture occurs:

$$A[f(x, y)/z] = \forall x(P(x, f(x, y)) \wedge \exists y Q(y)) \rightarrow P(y, f(x, y)).$$

Note that every ground term (not containing variables), in particular every constant symbol, is always free for substitution.

Renamings and substitutions in a formula

NB: renaming and substitution are *different* operations: renaming always acts on bound variables, while substitution always acts on free variables.

Also, renamings preserve the formula up to logical equivalence, while substitutions do not.

On the other hand, a suitable renaming of a formula can prepare it for a substitution, by rendering the term to be substituted free for such substitution in the renamed formula.

For instance, the term $f(x, y)$ is not free for substitution for y in

$$A = \forall x(P(x, y) \wedge \exists y Q(y)),$$

but it becomes free for such substitution after renaming of A to

$$A' = \forall x'(P(x', y) \wedge \exists y Q(y)).$$