# First Order Logic:
# Prenex normal form. Skolemization. Clausal form

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# Revision: CNF and DNF of propositional formulae

- A literal is a propositional variable or its negation.
- An elementary disjunction is a disjunction of literals.
  An elementary conjunction is a conjunction of literals.
- A disjunctive normal form (DNF) is a disjunction of elementary conjunctions.
- A conjunctive normal form (CNF) is a conjunction of elementary disjunctions.

# Conjunctive and disjunctive normal forms of first-order formulae

An *open* first-order formula is in disjunctive normal form (resp., conjunctive normal form) if it is a first-order instance of a propositional formula in DNF (resp. CNF), obtained by uniform substitution of atomic formulae for propositional variables.

Examples:

$$(\neg P(x) \vee Q(x,y)) \wedge (P(x) \vee \neg R(y))$$

is in CNF, as it is a first-order instance of $(\neg p \vee q) \wedge (p \vee \neg r)$;

$$(P(x) \wedge Q(x,y) \wedge R(y)) \vee \neg P(x)$$

is in DNF, as it is a first-order instance of $(\neg p \wedge q \wedge r) \vee \neg p$.

$$\forall x P(x) \vee Q(x,y)$$

and

$$\neg P(x) \vee (Q(x,y) \wedge R(y)) \wedge \neg R(y)$$

are not in either CNF or DNF.

# Prenex normal forms

A first-order formula $Q_1 x_1 ... Q_n x_n A$, where $Q_1, ..., Q_n$ are quantifiers and $A$ is an open formula, is in a prenex form.

The quantifier string $Q_1 x_1 ... Q_n x_n$ is called the prefix, and the formula $A$ is the matrix of the prenex form.

Examples:

$$\forall x \exists y (x > \mathbf{0} \rightarrow (y > \mathbf{0} \wedge x = y^2))$$

is in prenex form, while

$$\exists x (x = \mathbf{0}) \wedge \exists y (y < \mathbf{0})$$

and

$$\forall x (x > \mathbf{0} \vee \exists y (y > \mathbf{0} \wedge x = y^2))$$

are not in prenex form.

# Prenex conjunctive and disjunctive normal forms

If $A$ is in DNF then $Q_1x_1...Q_nx_nA$ is in prenex disjunctive normal form (PDNF); if $A$ is in CNF then $Q_1x_1...Q_nx_nA$ is in prenex conjunctive normal form (PCNF).

Examples:

$$\forall x \exists y(\neg x > \mathbf{0} \vee y > \mathbf{0})$$

is both in PDNF and in PCNF.

$$\forall x \exists y(\neg x > \mathbf{0} \vee (y > \mathbf{0} \wedge \neg x = y^2))$$

is in PDNF, but not in PCNF.

$$\forall x(x > \mathbf{0} \vee \exists y(y > \mathbf{0} \wedge x = y^2))$$

is neither in PCNF nor in PDNF.

# Transformation to prenex normal forms

THEOREM: Every first-order formula is equivalent to a formula in a prenex disjunctive normal form (PDNF) and to a formula in a prenex conjunctive normal form (PCNF).

Here is an algorithm:

1. Eliminate all occurrences of $\rightarrow$ and $\leftrightarrow$.
2. Import all negations inside all other logical connectives.
3. Use the equivalences:

$$(a) \quad \forall x P \wedge \forall x Q \equiv \forall x (P \wedge Q),$$

$$(b) \quad \exists x P \vee \exists x Q \equiv \exists x (P \vee Q),$$

to pull some quantifiers outwards and, after renaming the formula *whenever necessary*.

# Transformation to prenex normal forms cont'd

4. Use also the following equivalences, where $x$ does not occur free in $Q$ :

$$(c)\ \ \forall xP \wedge Q \equiv Q \wedge \forall xP \equiv \forall x(P \wedge Q),$$

$$(d)\ \ \forall xP \vee Q \equiv Q \vee \forall xP \equiv \forall x(P \vee Q),$$

$$(e)\ \ \exists xP \vee Q \equiv Q \vee \exists xP \equiv \exists x(P \vee Q),$$

$$(f)\ \ \exists xP \wedge Q \equiv Q \wedge \exists xP \equiv \exists x(P \wedge Q),$$

to pull all quantifiers in front and thus transform the formula into a prenex form.

5. Finally, transform the matrix in a DNF or CNF, just like a propositional formula.

# Transformation to prenex normal forms: example

$A = \exists z(\exists x Q(x,z) \vee \exists x P(x)) \rightarrow \neg(\neg\exists x P(x) \wedge \forall x \exists z Q(z,x))$.

1. Eliminating $\rightarrow$:

$A \equiv \neg\exists z(\exists x Q(x,z) \vee \exists x P(x)) \vee \neg(\neg\exists x P(x) \wedge \forall x \exists z Q(z,x))$

2. Importing the negation:

$A \equiv \forall z(\neg\exists x Q(x,z) \wedge \neg\exists x P(x)) \vee (\neg\neg\exists x P(x) \vee \neg\forall x \exists z Q(z,x))$

$\equiv \forall z(\forall x \neg Q(x,z) \wedge \forall x \neg P(x)) \vee (\exists x P(x) \vee \exists x \forall z \neg Q(z,x))$.

3. Using the equivalences (a) and (b):

$A \equiv \forall z \forall x(\neg Q(x,z) \wedge \neg P(x)) \vee \exists x(P(x) \vee \forall z \neg Q(z,x))$.

4. Renaming:

$A \equiv \forall z \forall x(\neg Q(x,z) \wedge \neg P(x)) \vee \exists y(P(y) \vee \forall w \neg Q(w,y))$.

5. Using the equivalences (c)-(f) to pull the quantifiers in front:

$A \equiv \forall z \forall x \exists y \forall w((\neg Q(x,z) \wedge \neg P(x)) \vee P(y) \vee \neg Q(w,y))$.

6. The resulting formula is in a prenex DNF.

For a prenex CNF we have to distribute the $\vee$ over $\wedge$:

$A \equiv \forall z \forall x \exists y \forall w((\neg Q(x,z) \vee P(y) \vee \neg Q(w,y)) \wedge (\neg P(x) \vee P(y) \vee \neg Q(w,y)))$.

# Skolemization I: Skolem constants

Skolemization: procedure for systematic elimination of the *existential quantifiers* in a first-order formula in a prenex form, by introducing new constant and functional symbols, called Skolem constants and Skolem functions, in the formula.

- *Simple case:* the result of Skolemization of the formula $\exists x \forall y \forall z A$ is the formula $\forall y \forall z A[c/x]$, where $c$ is a new (Skolem) constant.

▷▷ For instance, the result of Skolemization of the formula $\exists x \forall y \forall z (P(x, y) \rightarrow Q(x, z))$ is $\forall y \forall z (P(c, y) \rightarrow Q(c, z))$.

- More generally, the result of Skolemization of the formula $\exists x_1 \cdots \exists x_k \forall y_1 \cdots \forall y_n A$ is $\forall y_1 \cdots \forall y_n A[c_1/x_1, \ldots, c_k/x_k]$, where $c_1, \ldots, c_k$ are new (Skolem) constants.

Note that the resulting formula is not equivalent to the original one, but is equally satisfiable with it.

# Skolemization II: Skolem functions

- The result of Skolemization of $\forall y \exists z P(y, z)$ is $\forall y P(y, f(y))$, where $f$ is a new unary function, called Skolem function.

- More generally, the result of Skolemization of $\forall y \exists x_1 \cdots \exists x_k \forall y_1 \cdots \forall y_n A$ is $\forall y \forall y_1 \cdots \forall y_n A[f_1(y)/x_1, \ldots, f_k(y)/x_k]$, where $f_1, \ldots, f_k$ are new Skolem functions.

- The result of Skolemization of

$$\forall x \exists y \forall z \exists u A(x, y, z, u)$$

is

$$\forall x \forall z A[f(x)/y, g(x, z)/u),$$

where $f$ is a new unary Skolem function and $g$ is a new binary Skolem function.

Again, the resulting formula after Skolemization is not equivalent to the original one, but is equally satisfiable with it.

# Clausal form of first-order formulae

A literal is an atomic formula or a negation of an atomic formula.

Examples: $P(x)$, $\neg P(f(c, g(y)))$, $\neg Q\,(f(x, g(c)), g(g(g(y))))$.

A clause is a set of literals (representing their disjunction).

Example: $\{P(x), \neg P(f(c, g(y))), \neg Q(f(x, g(c)), g(g(g(y))))\}$

represents $P(x) \vee \neg P(f(c, g(y))) \vee \neg Q(f(x, g(c)), g(g(g(y))))$

All variables in a clause are assumed to be universally quantified.

A clausal form is a set of clauses (representing their conjunction).

Example:

$\{$
$\{P(x)\}$,
$\{\neg P(f(c)), \neg Q(g(x, x), y)\}$,
$\{\neg P(f(y)), P(f(c)), Q(y, f(x))\}$
$\}$.

# Transformation of first-order formulae to clausal form

THEOREM: Every first-order formula $A$ can be transformed to a clausal form $\{C_1, \ldots, C_k\}$ such that $A$ is equally satisfiable with the universal closure $\overline{(C_1 \wedge \cdots \wedge C_k)}$ of the conjunction of all clauses, considered as disjunctions.

The algorithm:

1. Transform $A$ to a prenex CNF.
2. Skolemize away all existential quantifiers.
3. Remove all universal quantifiers.
4. Write the matrix (which is in CNF) as a set of clauses.