Modal logics: an introduction

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Outline

- Non-classical logics in AI.
- Variety of modal logics. Brief historical remarks.
- Basic generic modal logic: syntax and Kripke semantics.
- Model checking in modal logic.
- Validity of modal formulae.
- Relationships between modal logic and first-order logic.

Reasoning with classical logic: pros and cons

Advantages:

- rich and uniform language for knowledge representation
- relatively simple syntax and well-understood semantics
- well-developed deductive systems and tools for automated reasoning

Disadvantages:

- cannot capture well some aspects natural language
- cannot capture adequately specific modes of reasoning
- algorithmic undecidability of logical consequence and validity

Non-classical logics

Non-classical logics include a wide variety of logical systems that restrict, extend, or modify the classical (propositional and first-order) logic, such as:

- Extensions of classical logic: modal logics.
- Extensions of classical logic: many-valued logics, fuzzy logics.
- Subsystems of classical logic: intuitionistic, linear, and other 'substructural' logics.
- Variations of classical logic: Relevant/relevance logics.
- Variations of classical logic: Non-monotonic logics.
- Variations of classical logic: Paraconsistent logics.
- etc.

In the rest of this course we will focus on modal logics.

They extend classical logic with additional logical connectives, called modal operators (or, modalities).

Modal logic: some historical remarks

- Aristotle: the 'Sea-battle tomorrow' argument.
  Necessary truths. Future truths.
- Medieval (modal) logic: mostly about theological issues.
- Leibniz: *A* is necessarily true if it is true in all possible worlds.
- C.I. Lewis: problems with the classical ('material') implication:
  - Irrelevance/non-causality: *If the Sun is hot, then 2+2=4.*
  - Ex falsum quodlibet:
    If 2+2=5 then the Moon is made of cheese.
  - Monotonicity:
    If I put sugar in my tea, then it will taste good.

    If I put sugar and I put petrol in my tea then it will taste good.
- Lewis' proposal: to add a strong implication

$$A \Rightarrow B := \Box(A \to B),$$

where $\Box X$ means '*X* is necessarily true'.

The emergence of modern modal logic

- Until the late 1950s: a collection of syntactic theories.
- The beginning of modern modal logic: in early 1960s with the introduction of the relational semantics by Saul Kripke.
- The philosophical idea behind the Kripke semantics is Leibniz' definition of necessary truth.
- Vigorous development of formal modal logic since the 1960s. A wide variety of modal systems, with different interpretations of the modal operators emerge.
- Gradually, modal logic changes focus and becomes increasingly popular as a versatile, suitably expressive, and computationally well-behaved framework for logical specification and reasoning in various areas of CS and AI.

Modes of truth.
Variety of modal reasoning and logics.

- Necessary and possible truths. Alletic logics.
- Truths over time. Temporal reasoning. Temporal logics.
- Reasoning about spatial relations. Spatial logics.
- Reasoning about ontologies. Description logics.
- Reasoning about knowledge. Epistemic logics.
- Reasoning about beliefs. Doxastic logics.
- Reasoning about obligations and permissions. Deontic logics.
- Reasoning about program executions. Logics of programs.
- Specification of transition systems. Logics of computations.
- Reasoning about many agents and their knowledge, beliefs, goals, actions, strategies, etc. Logics of multiagent systems.

What follows in the course?

- Basics of modal logics.
- Temporal logics of computations
- Epistemic and temporal-epistemic logics.
- Logics of multi-agent systems.

The basic propositional modal logic ML: syntax

Language of ML: logical connectives $\bot, \neg, \wedge$, and a unary modal operator $\Box$, and a set of atomic propositions $\mathrm{AP} = \{p_0, p_1, ...\}$.

Formulae:

$$\varphi = p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

Definable propositional connectives:

$\top := \neg\bot$;
$\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$;
$\varphi \rightarrow \psi := \neg(\varphi \wedge \neg\psi)$;
$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

The dual operator of $\Box$: $\Diamond\varphi = \neg\Box\neg\varphi$.

Meanings of the modal operators

- In aletic logic:
  $\Box\varphi$: 'φ is necessarily true'; $\Diamond\varphi$: 'φ is possibly true';
- In deontic logic:
  $\Box\varphi$: 'φ is obligatory'; $\Diamond\varphi$: 'φ is permitted';
- In logic of beliefs: $\Box\varphi$: 'the agent believes φ';
  $\Diamond\varphi$: 'the agent does not disbelieve φ';
- In logic of knowledge: $\Box\varphi$: 'the agent knows that φ';
  $\Diamond\varphi$: 'φ is consistent with the agent's knowledge';
- In temporal logic: $\Box\varphi$: 'φ will always be true',
  $\Diamond\varphi$: 'φ will become true sometime in the future ',
- In logic of (non-deterministic) programs:
  $\Box\varphi$: 'φ will be true after every execution of the program',
  $\Diamond\varphi$: 'φ will be true after some execution of the program'.

Some important modal axioms

- T: $\Box p \rightarrow p$;
- D: $\Box p \rightarrow \Diamond p$;
- B: $p \rightarrow \Box\Diamond p$;
- 4: $\Box p \rightarrow \Box\Box p$;
- 5: $\Diamond p \rightarrow \Box\Diamond p$;
- K: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$;

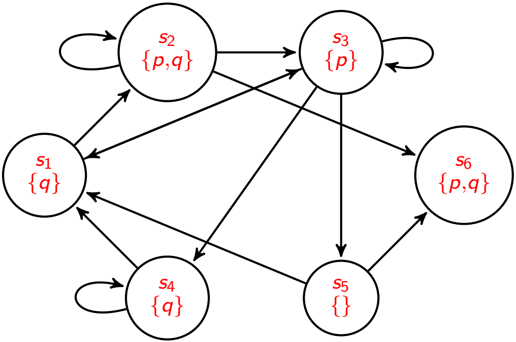
Semantic structures for modal logic

- Kripke frame: a pair (W, R), where:
  - W is a non-empty set of possible worlds ,
  - $R \subseteq W^2$ is an accessibility relation between possible worlds.

- Kripke model over a frame $\mathcal{T}$: a pair $(\mathcal{T}, V)$ where $V : \mathrm{AP} \to \mathcal{P}(W)$ is a valuation assigning to every atomic proposition the set of possible worlds where it is true.

  Sometimes, instead of valuations, Kripke models are defined in terms of labelling functions: $L : W \to \mathcal{P}(\mathrm{AP})$, where $L(s)$ comprises the atomic propositions true at the possible world s.

Kripke model: example



The valuation:
$V(p) = \{s_2, s_3, s_6\}, \quad V(q) = \{s_1, s_2, s_4, s_6\}.$

Kripke semantics of modal logic

Truth of a formula φ at a possible world u in a Kripke model $\mathcal{M} = (W, R, V)$, denoted $\mathcal{M}, u \models \varphi$, is defined as follows:

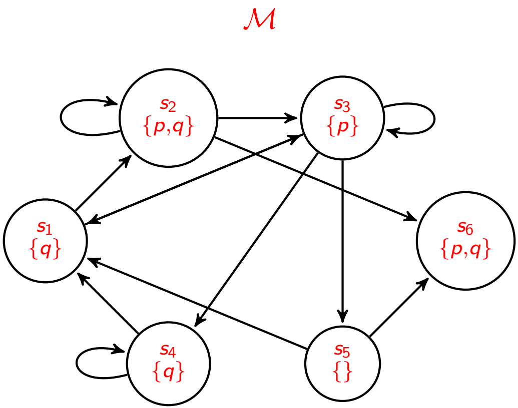
- $\mathcal{M}, u \models p$ iff $u \in V(p)$;
- $\mathcal{M}, u \not\models \bot$;
- $\mathcal{M}, u \models \neg\varphi$ iff $\mathcal{M}, u \not\models \varphi$;
- $\mathcal{M}, u \models \varphi_1 \wedge \varphi_2$ iff $\mathcal{M}, u \models \varphi_1$ and $\mathcal{M}, u \models \varphi_2$;
- $\mathcal{M}, u \models \Box\varphi$ iff $\mathcal{M}, w \models \varphi$ for *every* $w \in W$ such that Ruw.

Respectively,
$\mathcal{M}, u \models \Diamond\varphi$ if $\mathcal{M}, w \models \varphi$ for *some* $w \in W$ such that Ruw.

An important feature of modal logic: the notion of truth is local, i.e., at a state of a model.

However, modal formulae cannot refer explicitly to possible worlds.

Truth of modal formulae: exercises



Check the following:

$\mathcal{M}, s_1 \overset{?}{\models} q \wedge \Box p$. Yes.

$\mathcal{M}, s_1 \overset{?}{\models} \Box q$. No.

$\mathcal{M}, s_1 \overset{?}{\models} \Box\Diamond q$. Yes.

$\mathcal{M}, s_2 \overset{?}{\models} \Diamond(q \wedge \Box q)$.
Yes: take s_6.

$\mathcal{M}, s_2 \overset{?}{\models} \Box\Box(p \vee q)$.
No.

$\mathcal{M}, s_3 \overset{?}{\models} \Box(\neg q \rightarrow \Diamond\neg p)$. Yes;

$\mathcal{M}, s_4 \overset{?}{\models} \Diamond\Diamond\Box(q \wedge \neg p \wedge \Box q)$. Yes.

$\mathcal{M}, s_6 \overset{?}{\models} \Box(\Box q \rightarrow \Box\neg\Diamond(p \wedge \Box q))$. Yes.

Validity and satisfiability of modal formulae

A modal formula φ is:

- valid in a model $\mathcal{M}$, denoted $\mathcal{M} \models \varphi$, if it is true at every world of $\mathcal{M}$;
- valid at a possible world u in a frame $\mathcal{T}$, denoted $\mathcal{T}, u \models \varphi$, if it is true at u in every model on $\mathcal{T}$;
- valid in a frame $\mathcal{T}$, denoted $\mathcal{T} \models \varphi$, if it is valid on every model on $\mathcal{T}$;
- valid, denoted $\models \varphi$, if it is valid in every model (or frame).
- satisfiable, if it is true at some possible world of some model, i.e., if its negation is not valid.

Thus, modal logic can be used as a language to specify properties of Kripke models, and to specify properties of Kripke frames.

Extension of a formula

The extension of a formula φ in a Kripke model $\mathcal{M} = (W, R, V)$ is the set of states in $\mathcal{M}$ satisfying the formula:

$$\|\varphi\|_{\mathcal{M}} := \{s \mid \mathcal{M}, s \models \varphi\}.$$

The extension of a formula $\|\varphi\|_{\mathcal{M}}$ can be computed inductively on the construction of φ:

- $\|\bot\|_{\mathcal{M}} = \emptyset$;
- $\|p\|_{\mathcal{M}} = V(p)$
- $\|\neg\varphi\|_{\mathcal{M}} = W \setminus \|\varphi\|_{\mathcal{M}}$;
- $\|\varphi_1 \wedge \varphi_2\|_{\mathcal{M}} = \|\varphi_1\|_{\mathcal{M}} \cap \|\varphi_2\|_{\mathcal{M}}$;
- $\|\Box\varphi\|_{\mathcal{M}} = \{s \mid R(s) \subseteq \|\varphi\|_{\mathcal{M}}\}$.

Model checking of modal formulae

Model checking is a procedure checking whether a given model satisfies given property, usually specified in some logical language.

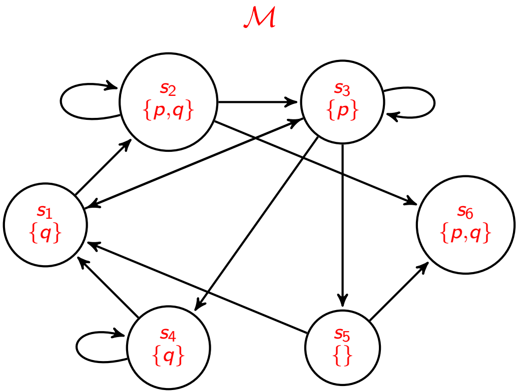
Model checking may, or may not, be algorithmically decidable, depending on the logical formalism and the class of models under consideration.

The main model checking problems for modal logic are:

1. Local model checking: given a Kripke model $\mathcal{M}$, a state $u \in \mathcal{M}$ and a modal formula φ, determine whether $\mathcal{M}, u \models \varphi$;
2. Global model checking: given a Kripke model $\mathcal{M}$ and a modal formula φ, determine the set $\|\varphi\|_{\mathcal{M}}$.

We are also interested in model satisfiability checking: given a Kripke model $\mathcal{M}$ and a formula φ, determine whether $\|\varphi\|_{\mathcal{M}} \neq \emptyset$.

Global model checking of modal formulae: exercises



Compute the following:

$\|\Box p\|_{\mathcal{M}} = \{s_1, s_2, s_6\}$.

$\|p \wedge \Box p\|_{\mathcal{M}} = \{s_2, s_6\}$.

$\|\Diamond(p \wedge \Box p)\|_{\mathcal{M}} =$
$\{s_1, s_2, s_5\}$.

$\|\neg q \rightarrow \Diamond(p \wedge \Box p)\|_{\mathcal{M}} =$
$\{s_1, s_2, s_4, s_5, s_6\}$.

$\|\Box\Box(\neg p \rightarrow q)\|_{\mathcal{M}} =$?

Global model checking of modal formulae: algorithm

Global model checking algorithm for ML: given a (finite) Kripke model $\mathcal{M}$ and a formula θ, compute the extensions $\|\varphi\|_{\mathcal{M}}$ for all subformulae φ of θ recursively, by labelling all possible worlds with those subformulae of θ that are true at those worlds, as follows:

- The labelling of atomic propositions is given by the valuation.
- The propositional cases are routine.
- $\|\Box\varphi\|$ consists of all states which have all their successors in $\|\varphi\|$, i.e. labelled by φ.

Question: What is the worst case complexity of global model checking of a modal formula φ is a given finite Kripke model $\mathcal{M}$

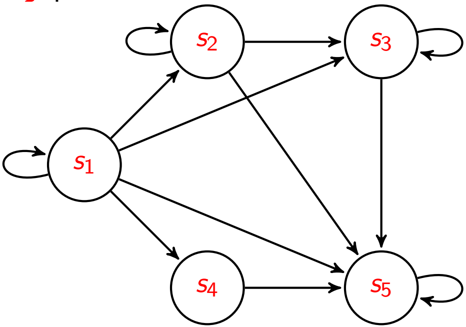
- in terms of the length of the formula $\|\varphi\|$?
- in terms of the size of the model $\|\mathcal{M}\|$?

Question: How can the procedure above be adapted efficiently to *local* model checking? What is the respective optimal complexity?

Validity of modal formulae in Kripke frames

Checking validity of a modal formula φ in a frame $\mathcal{T}$ requires checking validity of φ in all Kripke models based on $\mathcal{T}$, i.e., for all possible valuations of the atomic propositions occurring in φ.

$\mathcal{T}$:



Check the following:

$\mathcal{T}, s_1 \overset{?}{\models} \Box p \to p$. Yes.

$\mathcal{T}, s_1 \overset{?}{\models} p \to \Box\Diamond p$. No.

$\mathcal{T}, s_1 \overset{?}{\models} \Diamond\Diamond p \to \Diamond p$. Yes.

$\mathcal{T}, s_1 \overset{?}{\models} \Box p \to \Box\Box p$. Yes.

$\mathcal{M}, s_1 \overset{?}{\models} \Box(\Box p \to p)$. No.

$\mathcal{T} \overset{?}{\models} \Box p \to p$. No. $\mathcal{T} \overset{?}{\models} \Box p \to \Box\Box p$. Yes.

$\mathcal{T} \overset{?}{\models} \Box(p \to q) \to (\Box p \to \Box q)$. Yes.

Logical consequence of modal formulae

A modal formula φ is a logical consequence from a set of modal formulae Γ, denoted $\Gamma \models \varphi$, if for every Kripke model $\mathcal{M}$ and a world $w \in \mathcal{M}$:

$$\text{if } \mathcal{M}, w \models \Gamma \text{ then } \mathcal{M}, w \models \varphi.$$

Thus, every classical propositional logical consequence is a modal consequence, too.

Modal logical consequence has the basic properties of propositional logical consequence. In particular, the following are equivalent:

1. $\varphi_1, \ldots, \varphi_n \models \psi$.
2. $\varphi_1 \wedge \cdots \wedge \varphi_n \models \psi$.
3. $\models \varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \psi$.
4. $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_n \rightarrow \psi) \ldots)$.

Thus, logical consequence is reducible to validity in modal logic.

Besides: if $\varphi_1, \ldots, \varphi_n \models \psi$ then $\Box\varphi_1, \ldots, \Box\varphi_n \models \Box\psi$.

Validity of modal formulae

Some valid modal formulae:

- Every modal instance of a propositional tautology, i.e., every formula obtained by uniform substitution of modal formulae for propositional variables in a propositional tautology.

  For instance: $\Box p \vee \neg \Box p$; $(\Box p \wedge \Diamond \Box q) \to \Diamond \Box q$, etc.

- K: $\Box(p \to q) \to (\Box p \to \Box q)$;

- $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$.

- $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$.

- $\Box\varphi$, for every valid modal formula φ.

  E.g., $\Box(\Diamond p \vee \neg\Diamond p)$.

Unlike first-order logic, testing validity in modal logic is decidable, and PSPACE-complete.

Some properties of binary relations

A binary relation $R \subseteq X^2$ is called:

- reflexive if it satisfies $\forall x \, xRx$.
- irreflexive if it satisfies $\forall x \, \neg xRx$.
- serial if it satisfies $\forall x \exists y \, xRy$.
- functional if it satisfies $\forall x \exists ! y \, xRy$,
  where $\exists ! y$ means 'there exists a unique y'.
- symmetric if it satisfies $\forall x \forall y (xRy \rightarrow yRx)$.
- asymmetric if it satisfies $\forall x \forall y (xRy \rightarrow \neg yRx)$.
- antisymmetric if it satisfies $\forall x \forall y (xRy \wedge yRx \rightarrow x = y)$.
- connected if it satisfies $\forall x \forall y (xRy \vee yRx)$.
- transitive if it satisfies $\forall x \forall y \forall z ((xRy \wedge yRz) \rightarrow xRz)$.
- equivalence relation if it is reflexive, symmetric, and transitive.
- euclidean if it satisfies $\forall x \forall y \forall z ((xRy \wedge xRz) \rightarrow yRz)$.
- pre-order, (or quasi-order) if it is reflexive and transitive.
- partial order, if it is reflexive, transitive, and antisymmetric.
- linear order, (or total order) if it is a connected partial order.

Some relational properties of Kripke frames definable by modal formulae

Claim For every Kripke frame $\mathcal{T} = (W, R)$ the following holds:

- $\mathcal{T} \models \Box p \to p$ iff the relation R is reflexive.
- $\mathcal{T} \models \Box p \to \Diamond p$ iff the relation R is serial.

  Exercise: find a simpler modal formula that defines seriality.
- $\mathcal{T} \models p \to \Box\Diamond p$ iff $\mathcal{T} \models \Diamond\Box p \to p$ iff the relation R is symmetric.
- $\mathcal{T} \models \Box p \to \Box\Box p$ iff $\mathcal{T} \models \Diamond\Diamond p \to \Diamond p$ iff the relation R is transitive.
- $\mathcal{T} \models \Diamond p \to \Box\Diamond p$ iff $\mathcal{T} \models \Diamond\Box p \to \Box p$ iff the relation R is euclidean.

Standard translation of modal logic to first-order logic

L_0: a FO language with $=$, a binary predicate R, and individual variables $\mathrm{VAR} = \{x_0, x_1, ...\}$.

L_1: a FO language extending L_0 with a set of unary predicates $\{P_0, P_1, ...\}$, corresponding to the atomic propositions $p_0, p_1, ...$.

The formulae of ML are translated into L_1 by means of the following standard translation:

- $\mathrm{ST}(p_i; x) := P_i x$ for every $p_i \in \mathrm{AP}$;
- $\mathrm{ST}(\neg\phi; x) := \neg\mathrm{ST}(\phi; x)$.
- $\mathrm{ST}(\phi_1 \wedge \phi_2; x) := \mathrm{ST}(\phi_1; x) \wedge \mathrm{ST}(\phi_2; x)$;
- $ST(\Box\phi; x) := \forall y(Rxy \to \mathrm{ST}(\phi; y))$, where y is the first variable in VAR not used yet in the translation of $ST(\phi; x)$.
- Respectively, we obtain:
  $ST(\Diamond\phi; x) := \exists y(Rxy \wedge \mathrm{ST}(\phi; y))$, where y is the first variable in VAR not used yet in the translation of $ST(\phi; x)$.

  NB: It suffices to use, alternating, only two variables, x and y.

Every Kripke model $\mathcal{M} = (W, R, V)$ can be regarded as a FO structure for the language L_1, where the unary predicate P_i is interpreted as $V(p_i)$.

Now, for every Kripke model $\mathcal{M}, w \in \mathcal{M}$ and $\varphi \in \mathrm{ML}$:

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}, x := w \models_{\mathrm{FO}} \mathrm{ST}(\varphi; x),$$

Accordingly,

$$\mathcal{M} \models \varphi \text{ iff } \mathcal{M} \models_{\mathrm{FO}} \forall x \mathrm{ST}(\varphi; x).$$

Then, validity of a modal formula in a frame translates into:

$$\mathcal{T} \models \varphi \text{ iff } \mathcal{T} \models \forall P_1 \ldots \forall P_k \forall x \mathrm{ST}(\varphi; x).$$

where $P_1, \ldots, P_k$ are the unary predicates occurring in φ.

Thus, in terms of truth and validity in Kripke models, ML is a fragment of the first-order language L_1, while in terms of validity in Kripke frames, it is a fragment of universal monadic second order logic over L_0.

Some examples of standard translations of modal formulae to the FO language L_1

- $\mathrm{ST}(\Box p \to p; x) = \forall x_1 (Rxx_1 \to Px_1) \to Px$

  or, after renaming, just $\forall y (Rxy \to Py) \to Px$.

- $\mathrm{ST}(\Box \Diamond p; x) = \forall x_2 (Rxx_2 \to \exists x_1 (Rx_2 x_1 \wedge Px_1))$

  or, just $\forall y (Rxy \to \exists z (Ryz \wedge Pz))$.

  Note that this is equivalent to $\forall y (Rxy \to \exists x (Ryx \wedge Px))$.

- $\mathrm{ST}(\Box\Box\Diamond p; x) = \forall y (Rxy \to \forall z (Ryz \to \exists u (Rzu \wedge Pu)))$,

  equivalent to $\forall y (Rxy \to \forall x (Ryx \to \exists y (Rxy \wedge Py)))$.

On the correspondence between modal logic and first-order logic on Kripke frames

Not every modal formula defines a first-order property on Kripke frames.

For instance: for every Kripke frame $\mathcal{T} = (W, R)$ the following holds:
$\mathcal{T} \models \Box(\Box p \rightarrow p) \rightarrow \Box p$ iff the relation R is transitive and has no infinite increasing chains.
(NB The latter property is not definable in FOL.)

On the other hand, not every first-order definable property is definable by a modal formula.

For instance, the properties of irreflexivity, asymmetry, and connectiveness are not modally definable.

The correspondence between modal logic and first-order logic has been studied in depth.
See Handbook of Modal Logic for further details and references.

Modal logics defined semantically

Numerous important modal logics can be defined by restricting the class of Kripke frames in which modal formulae are interpreted.

For instance, such are the logics:

- The logic K of all Kripke frames.
- The logic D of all serial Kripke frames.
- The logic T of all reflexive Kripke frames.
- The logic B of all symmetric Kripke frames.
- The logic K4 of all transitive Kripke frames.
- The logic S4 of all reflexive and transitive Kripke frames.
- The logic S5 of all reflexive, transitive, and symmetric Kripke frames, i.e. all equivalence relations.

The basic modal logic K as a deductive system

A sound and complete axiomatic system for the basic modal logic **K** can be obtained by extending an axiomatic system for classical propositional logic **H** with the axiom

$$K: \quad \Box(p \to q) \to (\Box p \to \Box q)$$

and the Necessitation rule:

$$\frac{\varphi}{\Box\varphi}$$

Modal logics defined deductively

Many (but not all!) modal logics can be defined syntactically, as deductive systems, by extending **K** with additional axioms defining the respective class of Kripke frames.

- $\mathbf{T} = \mathbf{K} + \Box p \to p$;
- $\mathbf{D} = \mathbf{K} + \Box p \to \Diamond p$;
- $\mathbf{B} = \mathbf{K} + p \to \Box\Diamond p$;
- $\mathbf{K4} = \mathbf{K} + \Box p \to \Box\Box p$;
- $\mathbf{S4} = \mathbf{T4} = \mathbf{K4} + \Box p \to p$;
- $\mathbf{S5} = \mathbf{BS4} = \mathbf{S4} + p \to \Box\Diamond p$.

Deduction in modal logic

- Sound and complete axiomatic systems have been developed for many modal logics, including all those mentioned earlier.

  But, such systems are not suitable for practical deduction.

- Alternatively, resolution-based systems of deduction have been proposed, typically based on standard translation from modal logic to FOL.

  However, they only work well for some modal logics.

- Likewise, tableau-based systems of deduction have been developed for a range of modal logics. such systems are practically useful, but they all employ specific rules of inference to reflect their specific semantic properties.

- Also, systems of natural deduction and sequent calculi for various modal logics have been constructed.