

Exercises in Formal Proof

Martin Fränzle

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Exercise 1: Formal logic, properties of orders

Let $\text{even}(n)$ be the predicate telling whether the natural number n is even. Which of the following statements are true?

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| 1. $\forall n \in \mathbf{N}.(\text{even}(n) \vee \neg \text{even}(n)),$ | 2. $(\forall n \in \mathbf{N}.\text{even}(n)) \vee (\forall n \in \mathbf{N}.\neg \text{even}(n)),$ |
| 3. $\forall n \in \mathbf{N}.\exists m \in \mathbf{N}.m > n,$ | 4. $\exists m \in \mathbf{N}.\forall n \in \mathbf{N}.m > n,$ |
| 5. $\forall n \in \mathbf{N}.\exists m \in \mathbf{N}.m < n,$ | 6. $\forall n \in \mathbf{R}_{>0}.\exists m \in \mathbf{R}_{>0}.m < n.$ |

Here, $\mathbf{R}_{>0}$ denotes the set $\{x \in \mathbf{R} \mid x > 0\}$ of strictly positive real numbers.

Exercise 2: Set equality

Show that the two sets $\{n - m \mid n, m \in \mathbf{N}\}$ and $\mathbf{N} \cup \{-k \mid k \in \mathbf{N}\}$ are equal.

Exercise 3: Induction on natural numbers

1. Prove by induction that

$$\sum_{i=1}^n \frac{2i-1}{2} = \frac{n^2}{2}$$

holds for each $n \in \mathbf{N}$.

2. Show by induction that any integer amount of postage greater than or equal to 2 DKK can be built using only 2 DKK and 3 DKK stamps.

Exercise 4: Induction on lists/strings

A *palindrome* is a string that reads the same if read from left to right or the other way round. I.e., the string $\langle w_1, w_2, \dots, w_k \rangle$ is a palindrome iff $w_1 = w_k$ holds, $w_2 = w_{k-1}$ holds, and so forth.

Let $w = \langle w_1, \dots, w_k \rangle$ with $k \geq 1$ be a string over $\{ 'a', \dots, 'z' \}$. Show that if both w and its tail¹ $\langle w_2, \dots, w_k \rangle$ are palindromes then $\forall i \leq k. w_i = w_1$ holds, i.e. w repeats just one letter.

(Hint: You can use induction with respect to the prefix order on strings, together with the fact that $\langle w_1, \dots, w_k \rangle$ being a palindrome implies that $\langle w_2, \dots, w_{k-1} \rangle$ also is a palindrome.)

¹In case $k = 1$, the tail $\langle w_2, \dots, w_k \rangle$ is the empty string, which trivially is a palindrome.

Exercise 5: “Goats and elephants don’t coexist”

Do you believe in the well-known fact that any finite assembly of goats and elephants does either contain only goats or only elephants, which is expressed by the following theorem? If not then explain what is wrong with its inductive proof.

Theorem 1 *Let U be a nonempty set of animals (U is the universe of all goats and elephants in the world), and let $f : U \rightarrow \{\text{goat}, \text{elephant}\}$ be the function identifying the species of the individual animals in U .*

Let $S \subset U$ be an arbitrary finite nonempty set of animals (e.g. the collection of a particular zoological garden). Then either $\forall x \in S. f(x) = \text{goat}$ or $\forall x \in S. f(x) = \text{elephant}$, i.e. S contains either only goats or only elephants.

Proof: By induction on the cardinality of S :

Induction basis, $|S| = 1$: If $|S| = 1$ then S contains exactly one element x , and trivially either $f(x) = \text{goat}$ or $f(x) = \text{elephant}$. Hence, S contains either only goats or only elephants, which shows the proposition in the base case. $\square$

Induction step, $|S| = k > 1$: We have to show that the premise

$$\forall S' \subset U \text{ with } |S'| < k. (\forall x \in S'. f(x) = \text{goat} \vee \forall x \in S'. f(x) = \text{elephant})$$

implies $(\forall x \in S. f(x) = \text{goat}) \vee (\forall x \in S. f(x) = \text{elephant})$.

Therefore, we start from the assumption that each subset $S' \subset U$ with $|S'| < k$ satisfies either $\forall x \in S'. f(x) = \text{goat}$ or $\forall x \in S'. f(x) = \text{elephant}$ (the “induction hypothesis”). We will show that this implies $(\forall x \in S. f(x) = \text{goat}) \vee (\forall x \in S. f(x) = \text{elephant})$ (the proposition).

We start by distinguishing two cases:

1. $\forall x \in S. f(x) = \text{elephant}$: In this case, the proposition is trivially satisfied. $\square$
2. There exists $y \in S$ such that $f(y) \neq \text{elephant}$: We construct a new set $S_1 \subset S$ by removing an arbitrary element $z \neq y$ from S , i.e. $S_1 = S \setminus \{z\}$ for some $z \in S \setminus \{y\}$. Such a z exists as $|S| = k > 1$.

By construction of S_1 , we know that $|S_1| = |S| - 1 = k - 1$ — which makes the induction premise applicable — and that $y \in S_1$ with $f(y) \neq \text{elephant}$, implying $\neg(\forall x \in S_1. f(x) = \text{elephant})$. Using the induction hypothesis, this implies $\forall x \in S_1. f(x) = \text{goat}$. Hence, as $S_1 = S \setminus \{z\}$, the only possible non-goat in S could be z .

However, we can now exchange z and y by defining $S_2 = S \setminus \{y\}$. Now, all elements of S_2 except perhaps z are goats, because $S_2 \setminus S_1 = \{z\}$ and we know that $\forall x \in S_1. f(x) = \text{goat}$. As $|S_2| = |S| - 1 = k - 1$, we can again apply the induction hypothesis, which yields that either all elements of S_2 are elephants or all are goats. Because all elements of S_2 except z are already known to be goats, this implies that $\forall x \in S_2. f(x) = \text{goat}$. In particular, as $z \in S_2$, we obtain $f(z) = \text{goat}$.

Together with the previously proven $\forall x \in S_1. f(x) = \text{goat}$, this implies $\forall x \in S. f(x) = \text{goat}$, as $S_1 = S \setminus \{z\}$. Hence, the proposition holds. $\square$

Exercise 6: Well-founded orders

Provide total well-founded orders for the following sets:

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|--|----------------|---------------------|
| 1. $\{p \in \mathbb{N} \mid p \text{ is prime}\},$ | 2. $\{a\}^*,$ | 3. $\{0, 1, 2\}^*,$ |
| 4. $\mathbb{N} \times \{0, 1, 2\}^*,$ | 5. $\{4711\},$ | 6. $\mathbb{Z}.$ |

Are the total well-founded orders unique? If not: is there a natural choice?

Exercise 7: Induction vs. order

Take the non-negative real numbers $\mathbf{R}_{\geq 0}$ with their native order $<$.

1. Is $<$ a well-founded order on the non-negative reals?
2. Is the induction scheme

Induction basis: Show $P(x)$ for all minimal elements $x \in S$, and

Induction step: show that $\forall y < x. P(y)$ implies $P(x)$ for arbitrary $x \in S$, then

Conclusion: $P(x)$ for each $x \in S$.

sound² on $S = (\mathbf{R}_{\geq 0}, <)$?

If not then please provide an example of an obviously wrong property which can be “proven” by above induction scheme. (*Hint:* You may try with a property you know to be true of all natural numbers, yet false on some (or most) non-negative reals.)

²A verification scheme is sound if all properties provable by the scheme are actually true.