Balanced Search Trees

2-3-4 trees
red-black trees

References: Algorithms in Java (handout)

Balanced search trees

Dynamic sets

- Search
- Insert
- Delete
- Maximum
- Minimum
- Successor(x) *(find minimum element ≥ x)*
- Predecessor(x) *(find maximum element ≤ x)*

This lecture: 2-3-4 trees, red-black trees

Next time: Tiered vektor (not a binary search tree, but maintains a dynamic set).

In two weeks time: Splay trees

Dynamic set implementations

Worst case running times

Implementation	search	insert	delete	minimum	maximum	successor	predecessor
linked lists	O(n)	O(1)	O(1)	O(n)	O(n)	O(n)	O(n)
ordered array	O(log n)	O(n)	O(n)	O(1)	O(1)	O(log n)	O(log n)
BST	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)

In worst case h=n.

In best case h= log n (fully balanced binary tree)

Today: How to keep the trees balanced.

2-3-4 trees

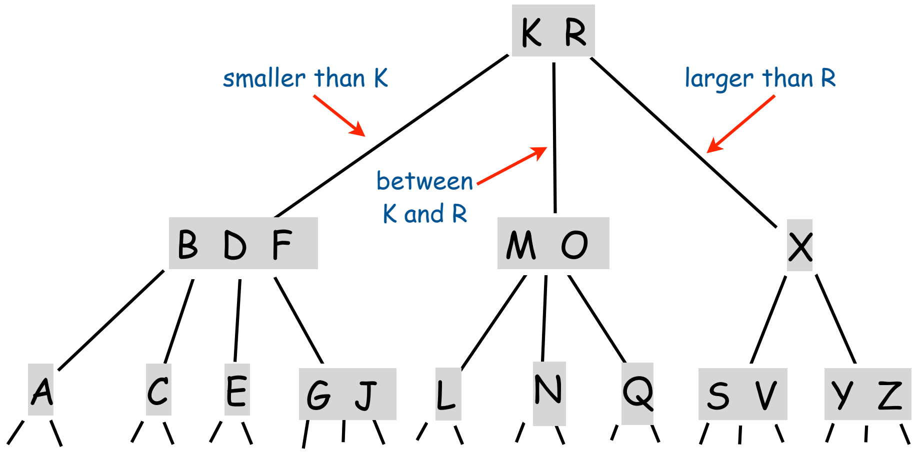
2-3-4 trees

2-3-4 trees. Allow nodes to have multiple keys.

Perfect balance. Every path from root to leaf has same length.

Allow 1, 2, or 3 keys per node

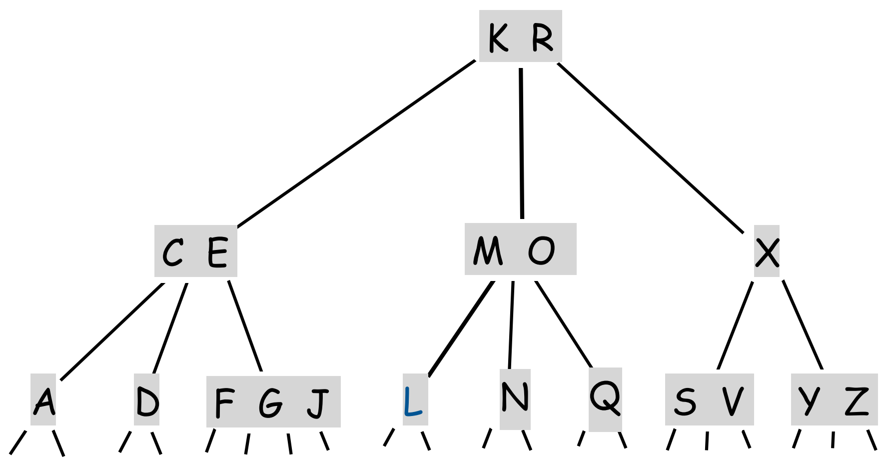
- 2-node: one key, 2 children
- 3-node: 2 keys, 3 children
- 4-node: 3 keys, 4 children



Searching in a 2-3-4 tree

Search.

- Compare search key against keys in node.
- Find interval containing search key
- Follow associated link (recursively)



Searching in a 2-3-4 tree

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- Find interval containing search key
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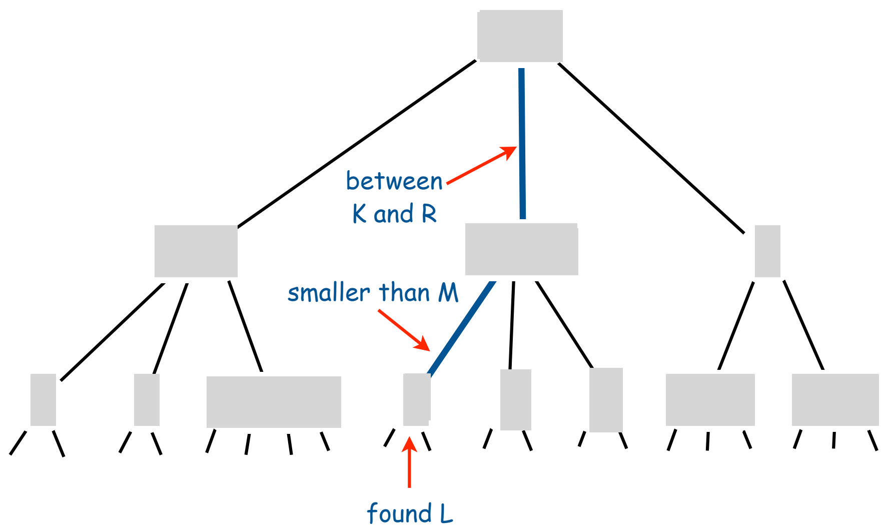
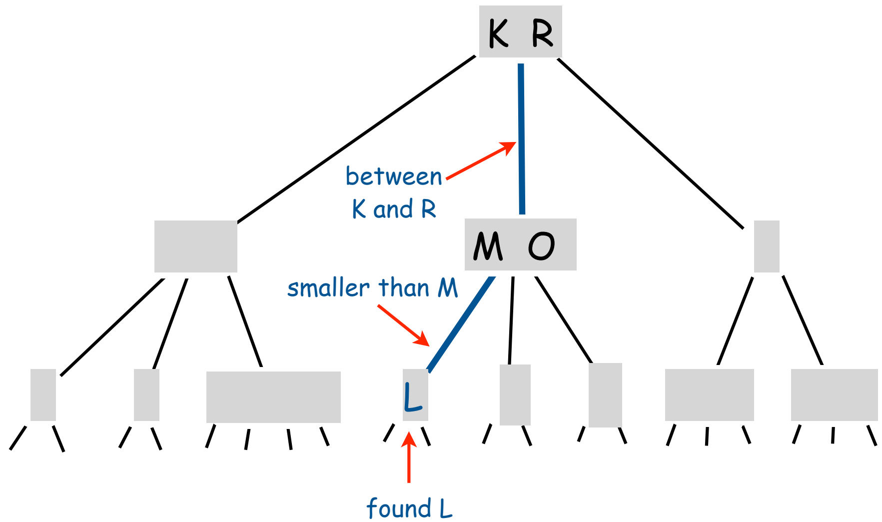
Ex. Search for L



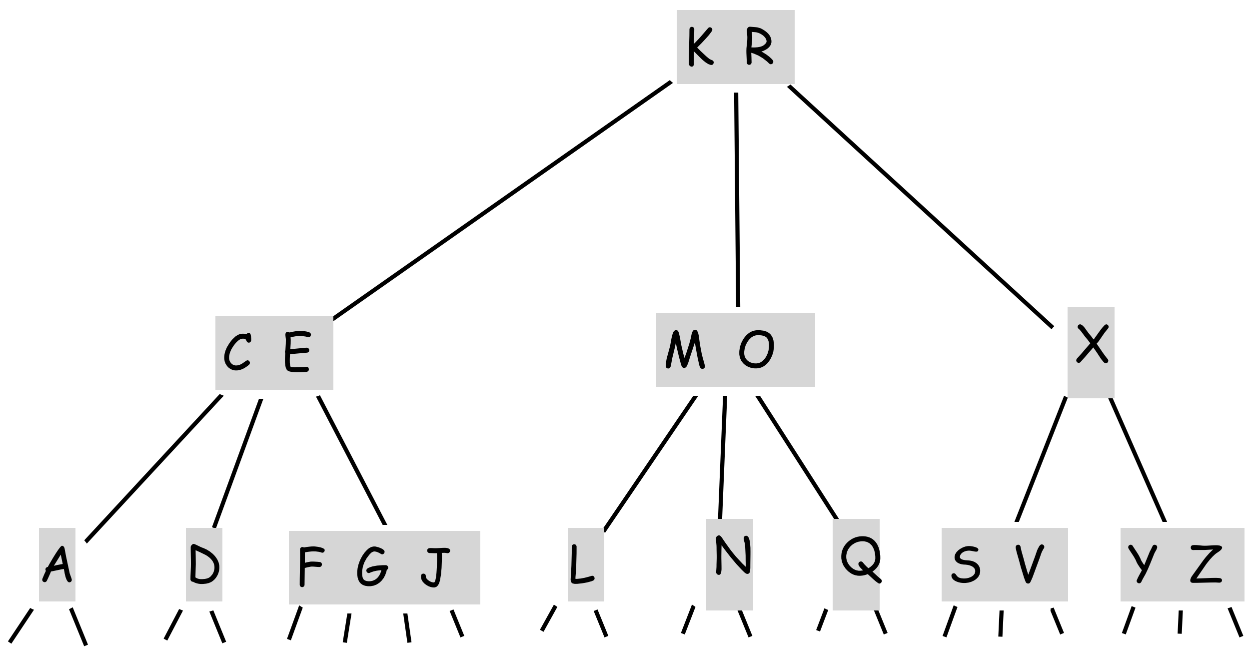
Predecessor and successor in a 2-3-4 tree

Where is the predecessor of L?

And the successor of L?



Insertion in a 2-3-4 tree

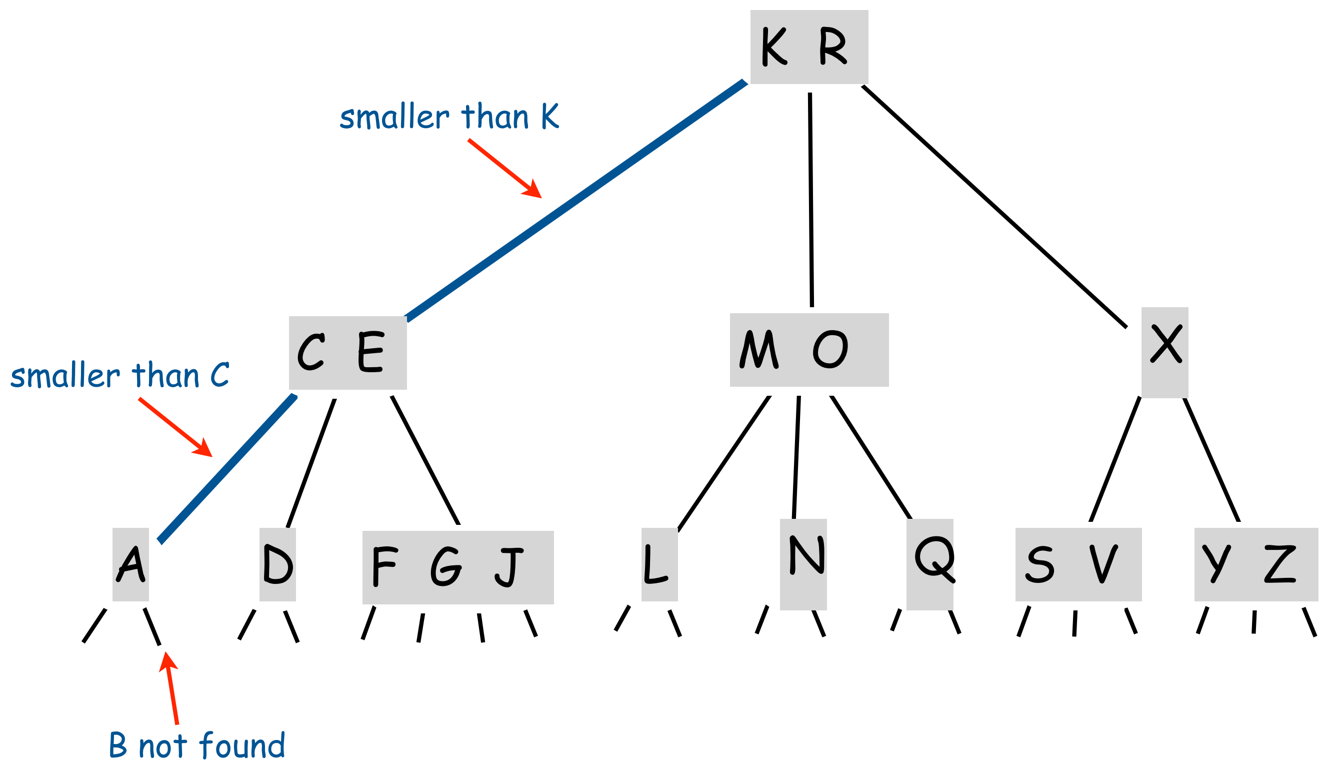


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.

Ex. Insert B

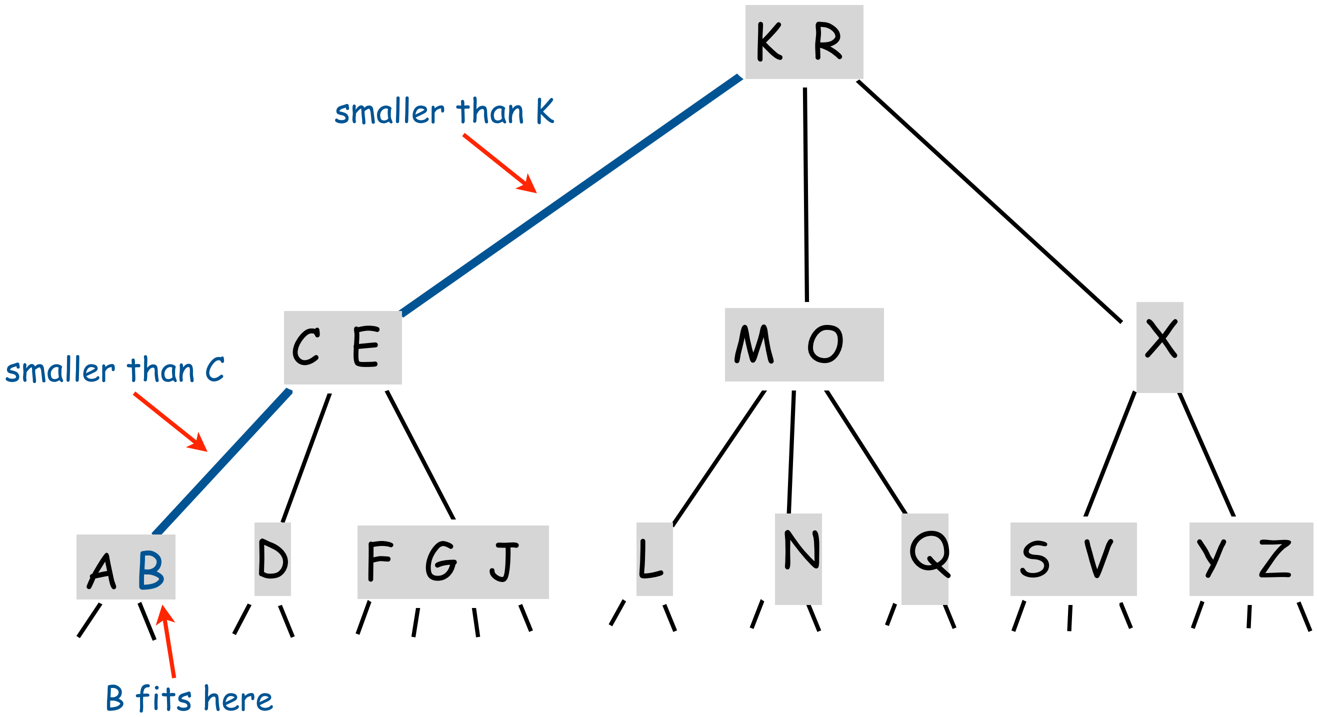


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node

Ex. Insert B

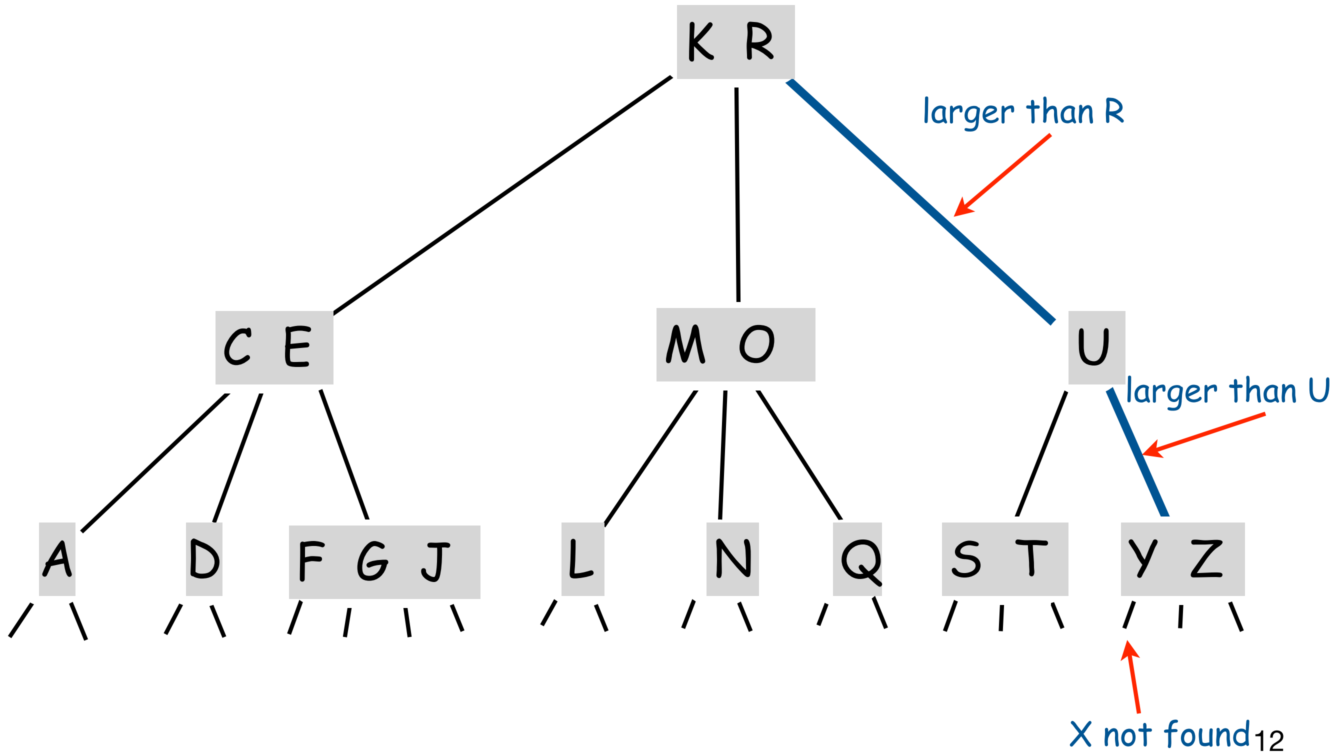


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node

Ex. Insert X

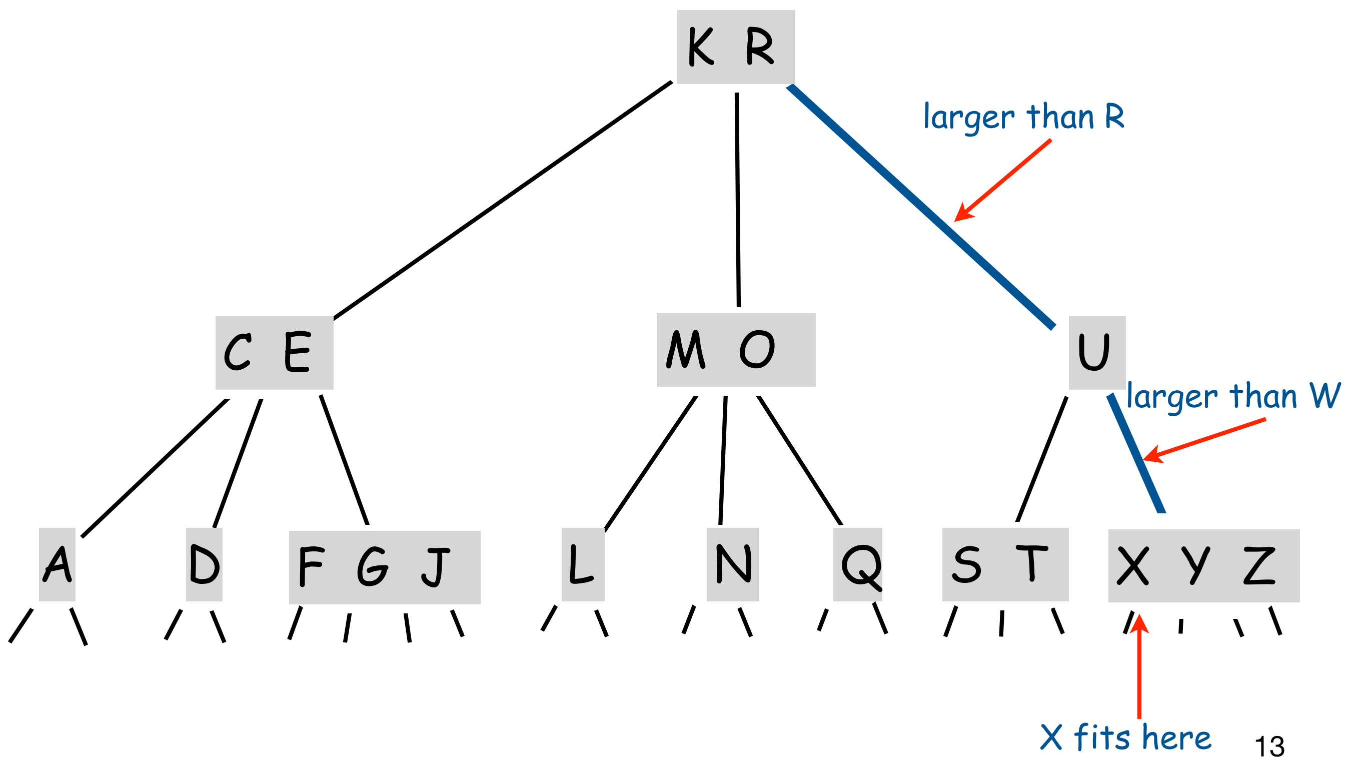


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node
- 3-node at bottom: convert to 4-node

Ex. Insert X



Insertion in a 2-3-4 tree

Insert.

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- 2-node at bottom: convert to 3-node
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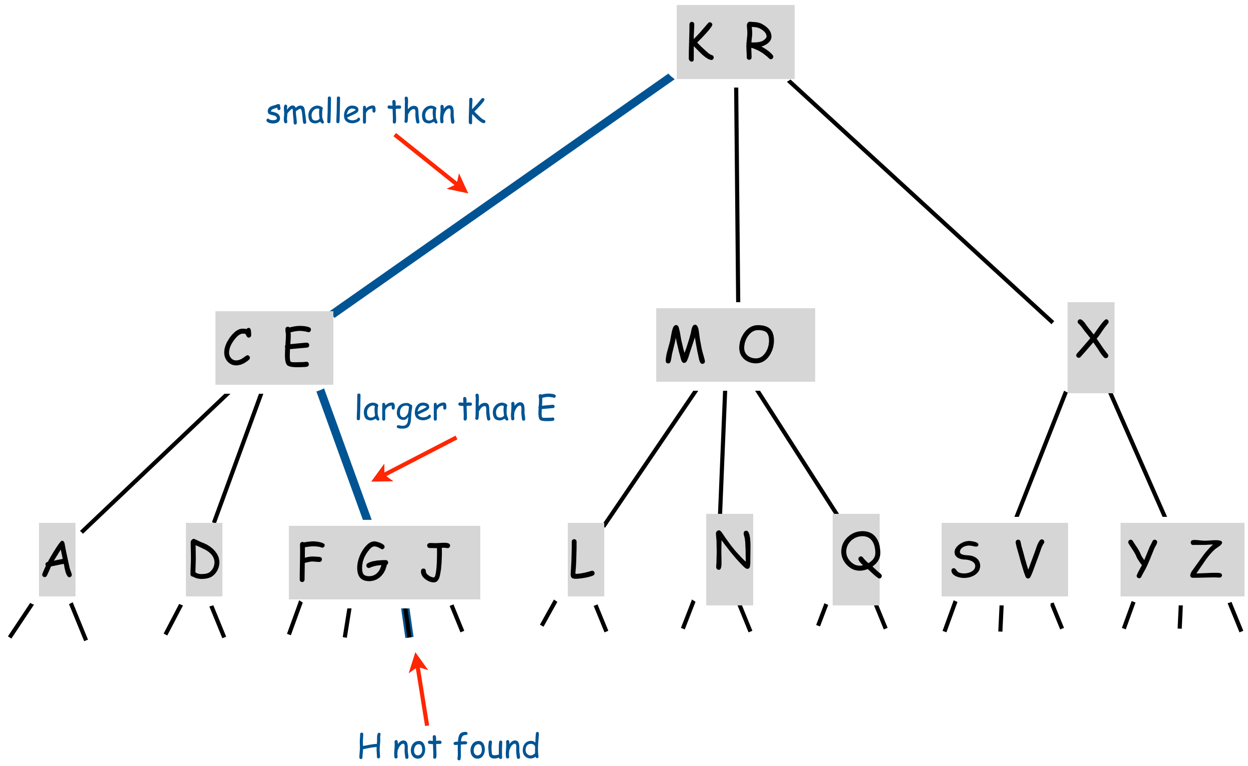
Ex. Insert H

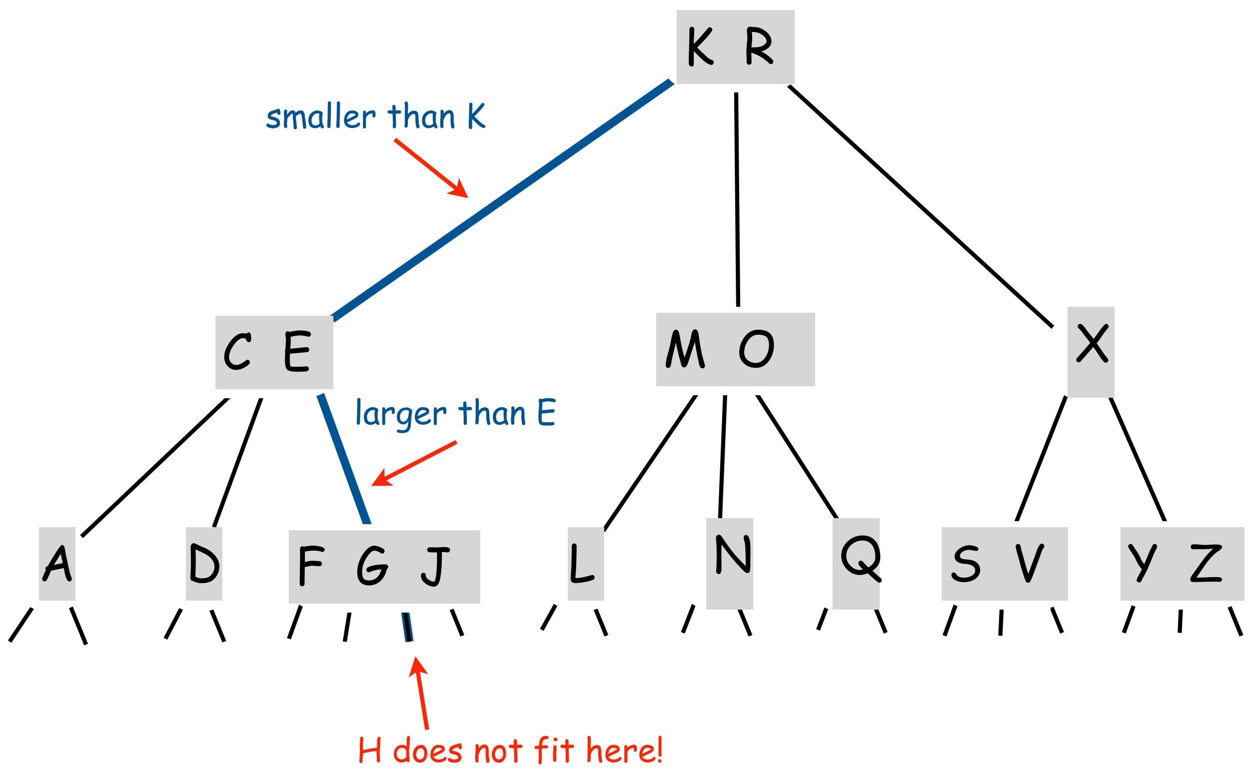


Insertion in a 2-3-4 tree

Insert.

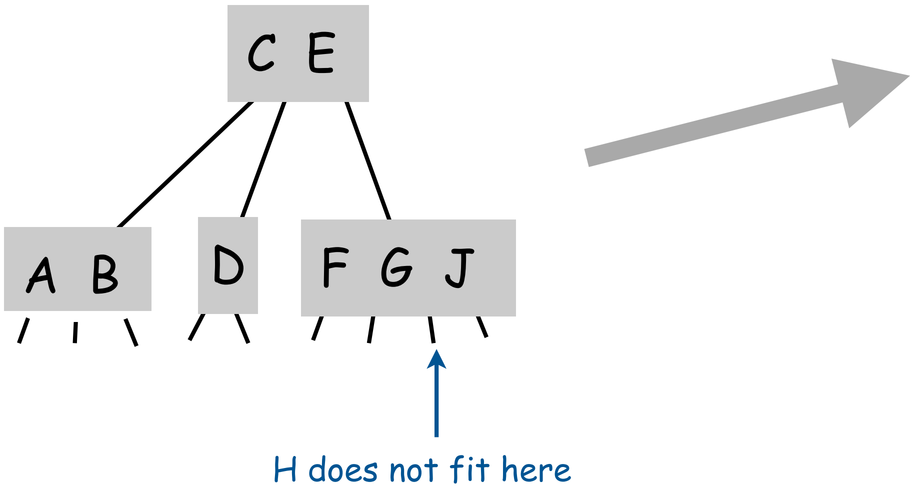
- Search to bottom for key.
- 2-node at bottom: convert to 3-node
- 3-node at bottom: convert to 4-node
- 4-node at bottom: ??

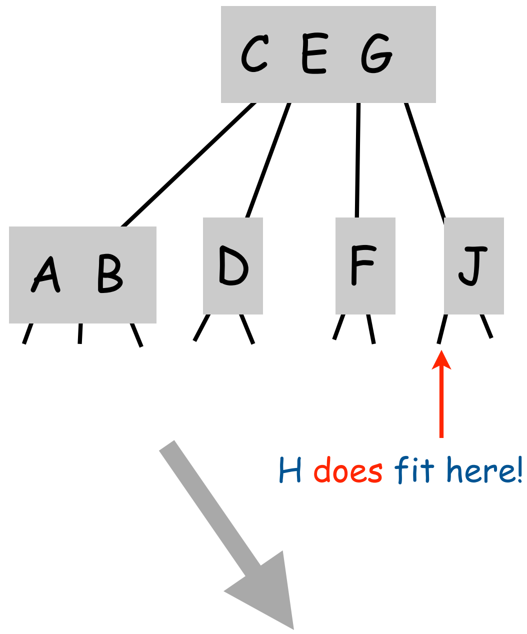
Ex. Insert H

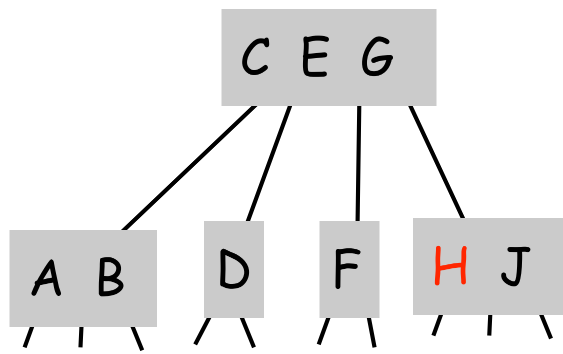


Splitting a 4-node in a 2-3-4 tree

Idea: split the 4-node to make room







Problem: Doesn't work if parent is a 4-node

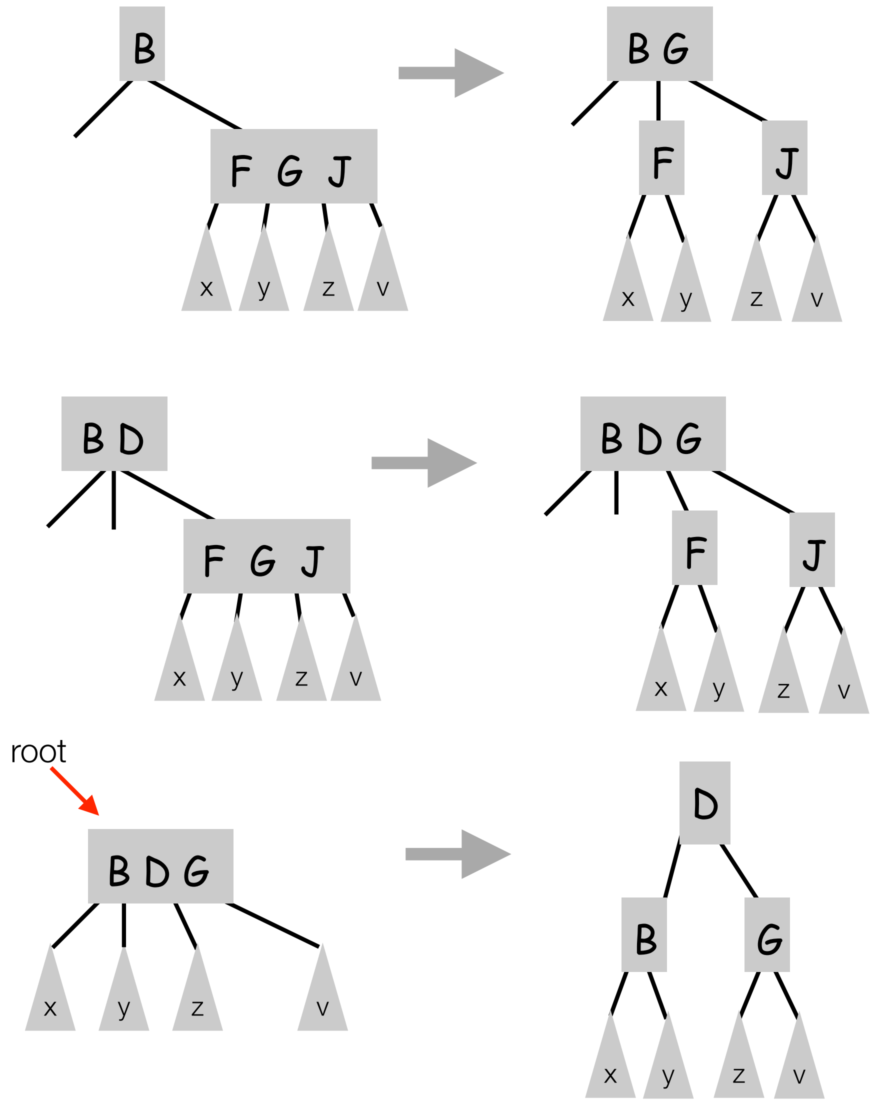
Solution 1: Split the parent (and continue splitting while necessary).

Solution 2: Split 4-nodes on the way down.

Splitting 4-nodes in a 2-3-4 tree

Idea: split 4-nodes on the way down the tree.

- Ensures last node is not a 4-node.
- Transformations to split 4-nodes:



Invariant. Current node is not a 4-node.

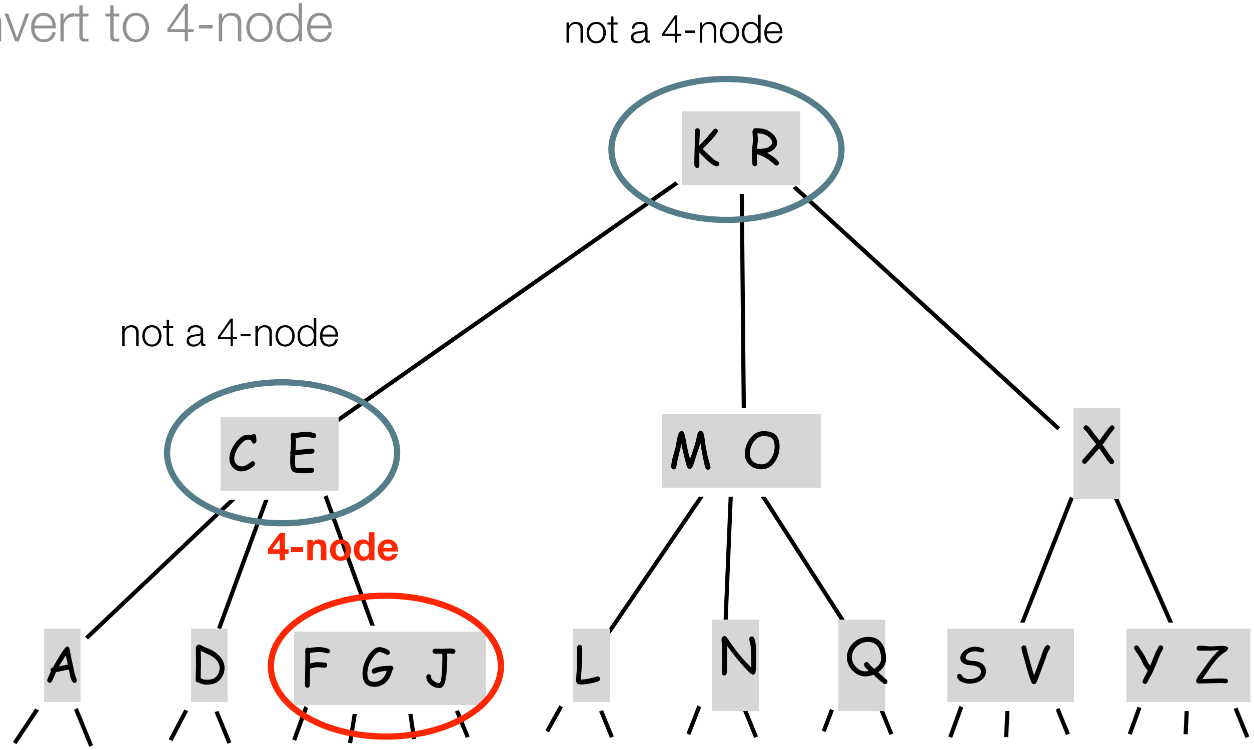
Consequence. Insertion at bottom is easy since it's not a 4-node.

Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node
- 3-node at bottom: convert to 4-node
- 4-node at bottom: ??

Ex. Insert H



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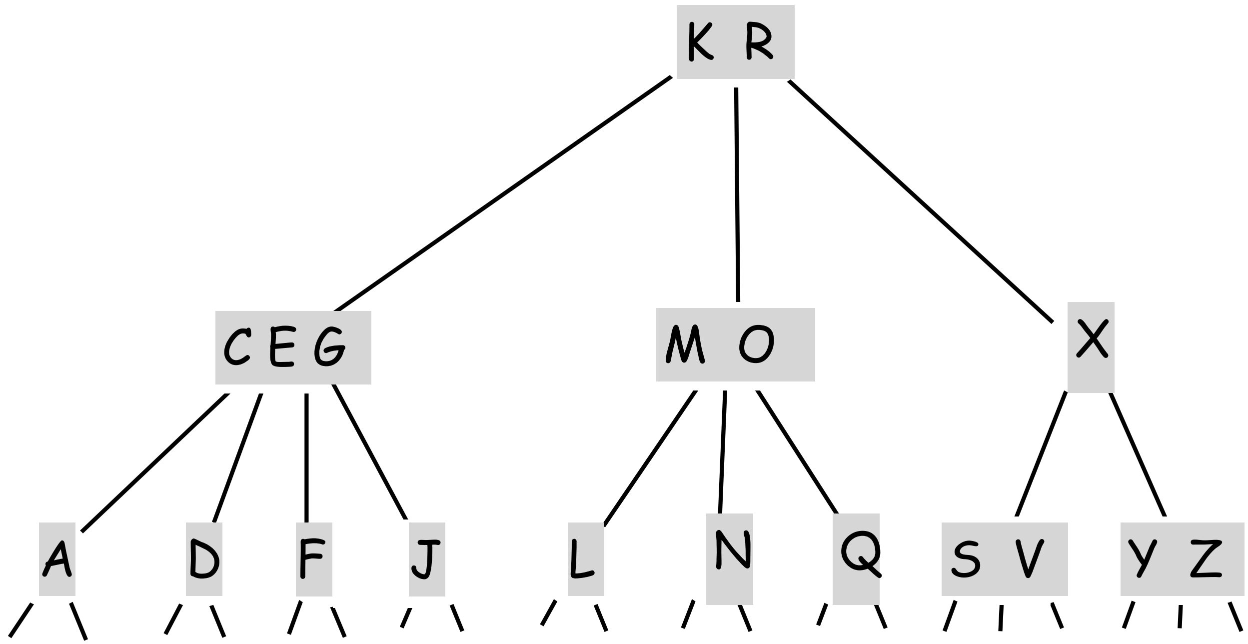

Insertion in a 2-3-4 tree

Insert.

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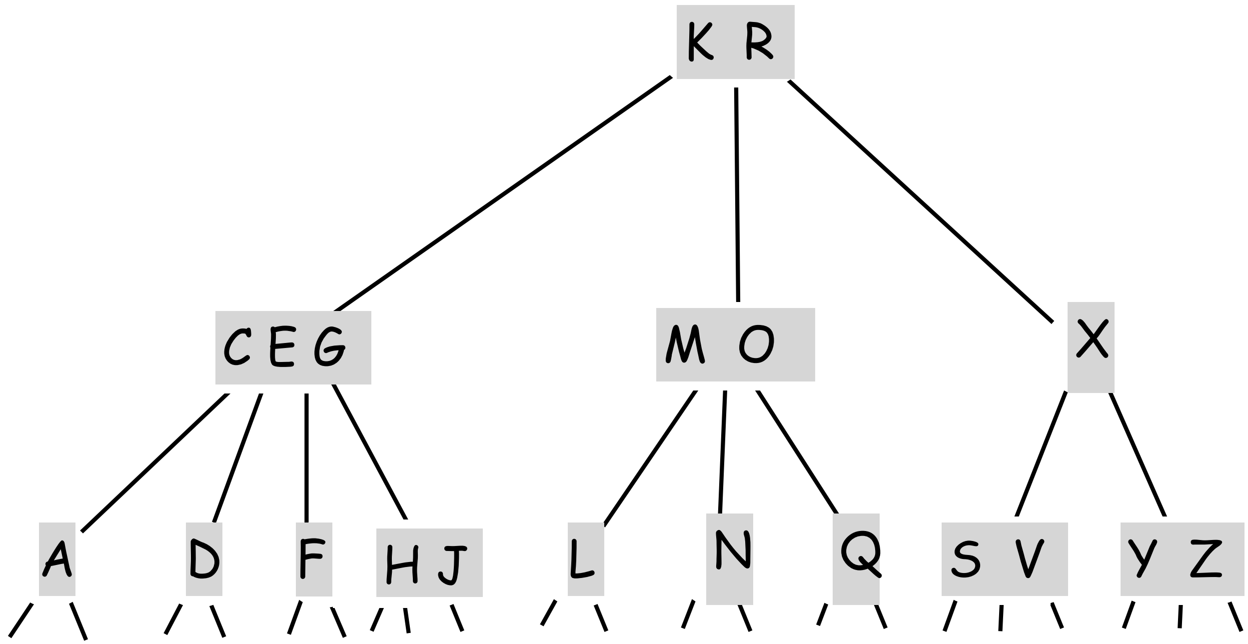
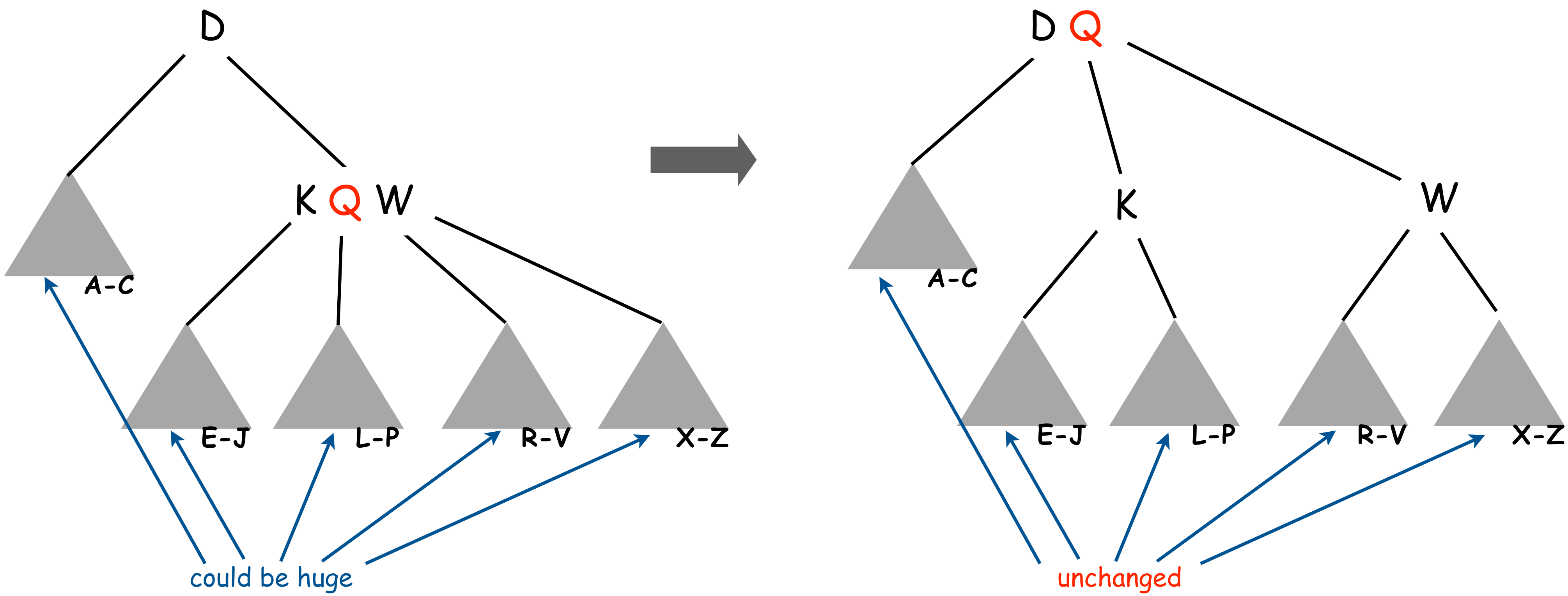
Ex. Insert H



Splitting 4-nodes in a 2-3-4 tree

Local transformations that work anywhere in the tree.

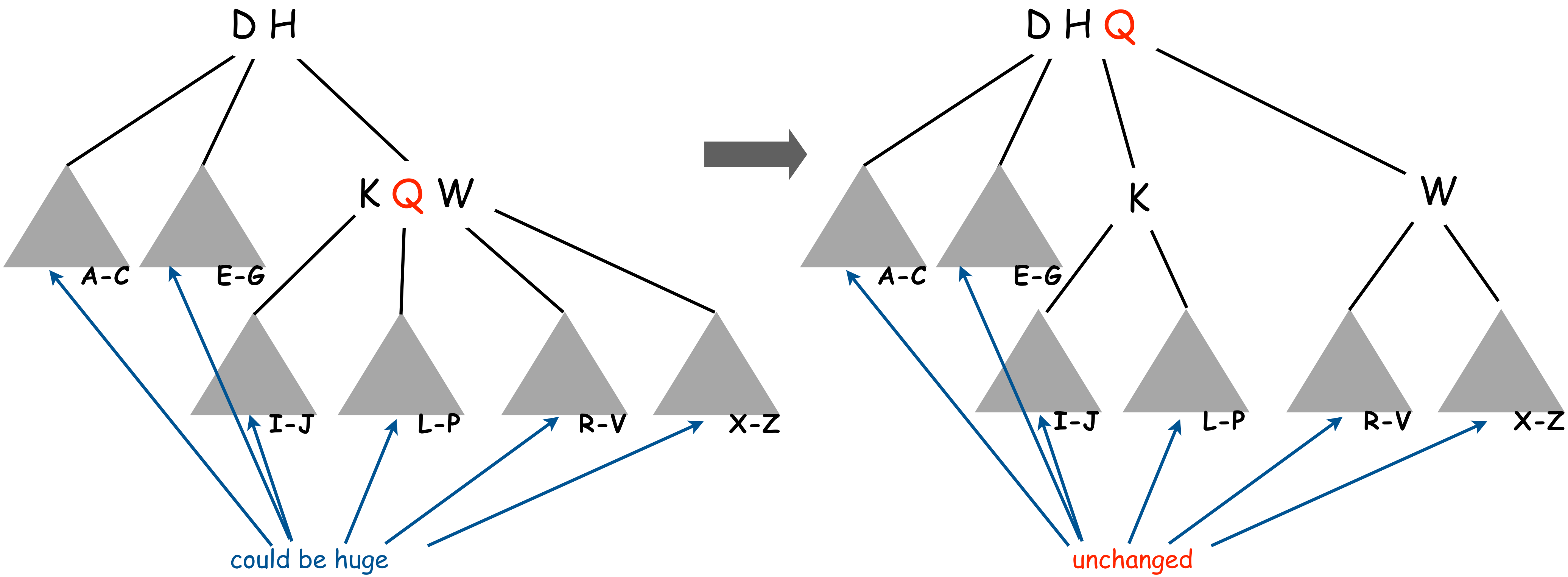
Ex. Splitting a 4-node attached to a 2-node



Splitting 4-nodes in a 2-3-4 tree

Local transformations that work anywhere in the tree

Ex. Splitting a 4-node attached to a 3-node



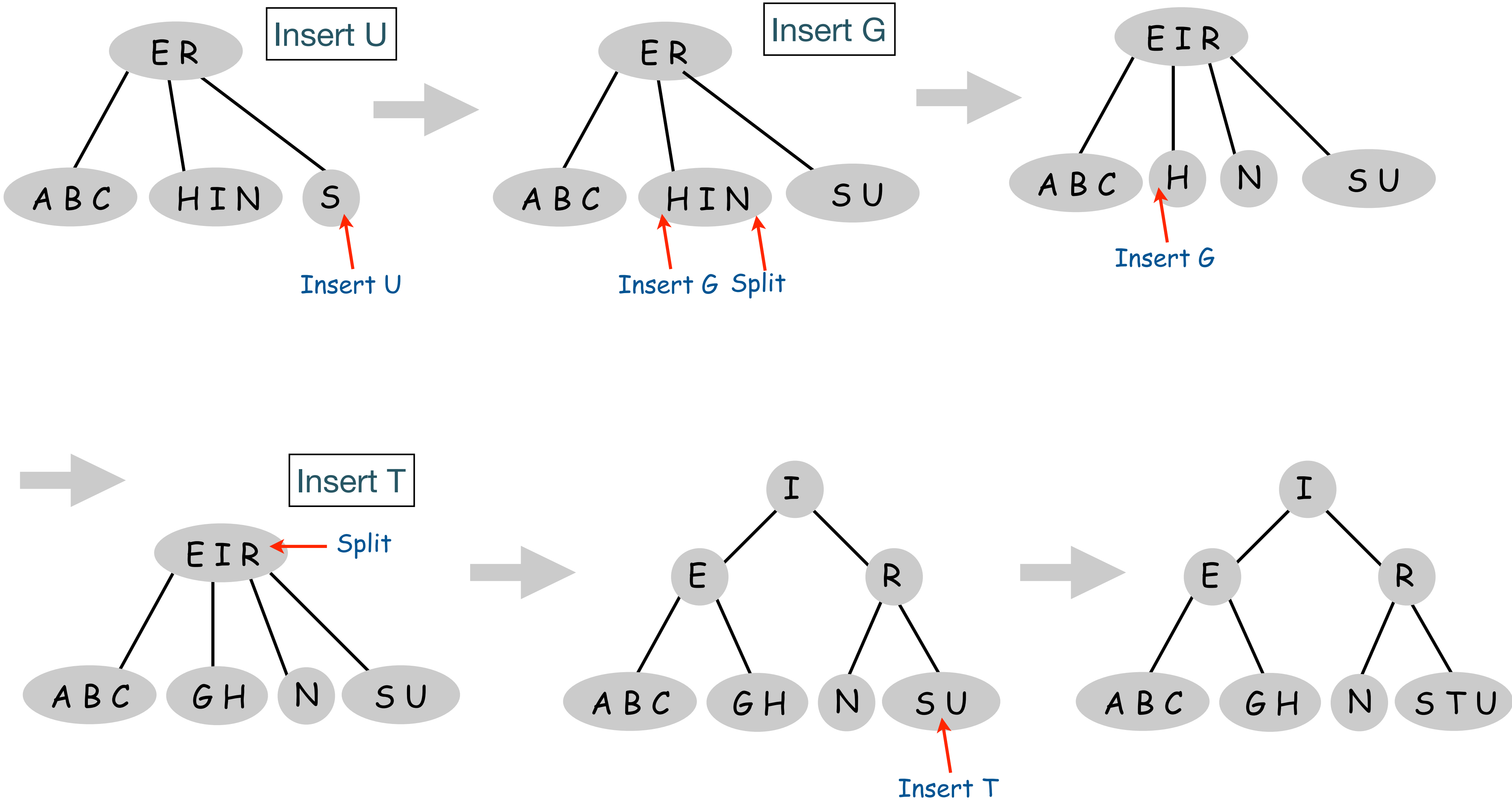
Splitting 4-nodes in a 2-3-4 tree

Local transformations that work anywhere in the tree.

Splitting a 4-node attached to a 4-node never happens when we split nodes on the way down the tree.

Invariant. Current node is not a 4-node.

Insertion 2-3-4 trees

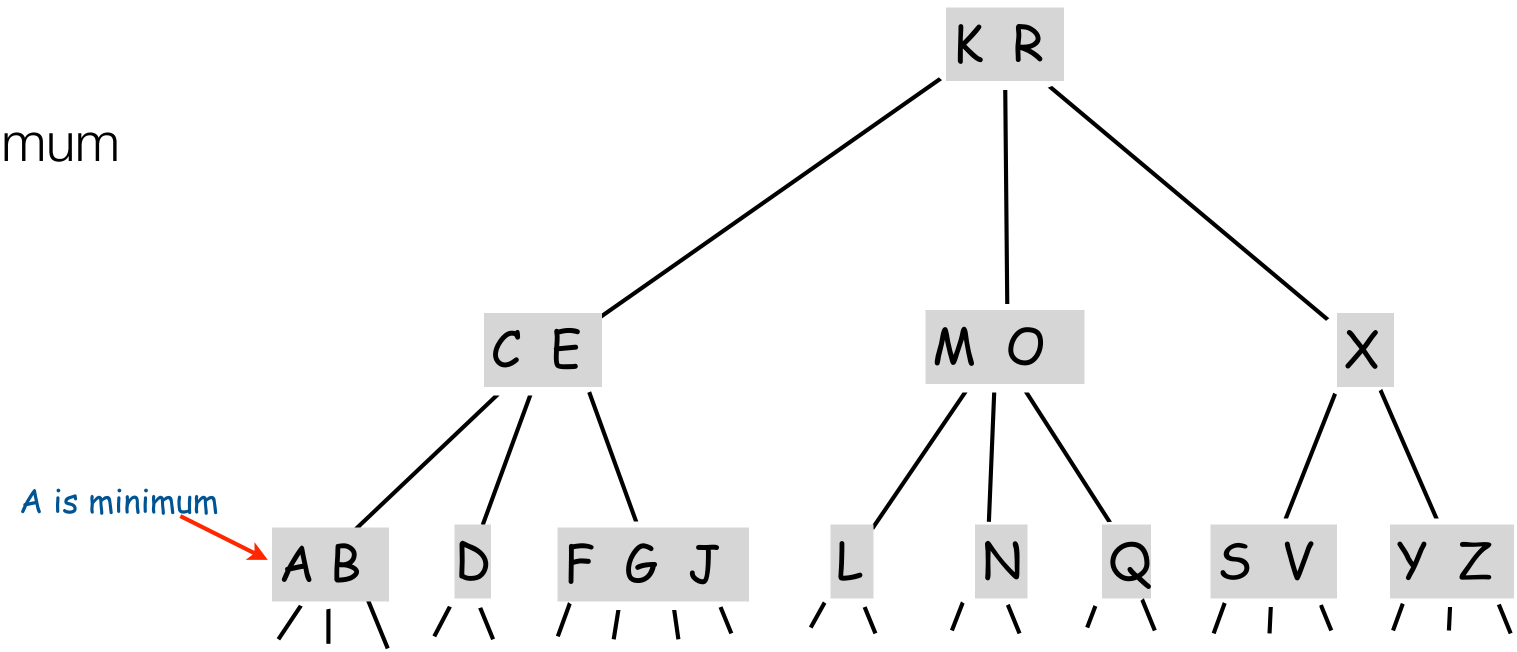


Deletions in 2-3-4 trees

Delete minimum:

- minimum always in leftmost leaf
- If 3- or 4-node: delete key

Ex. Delete minimum



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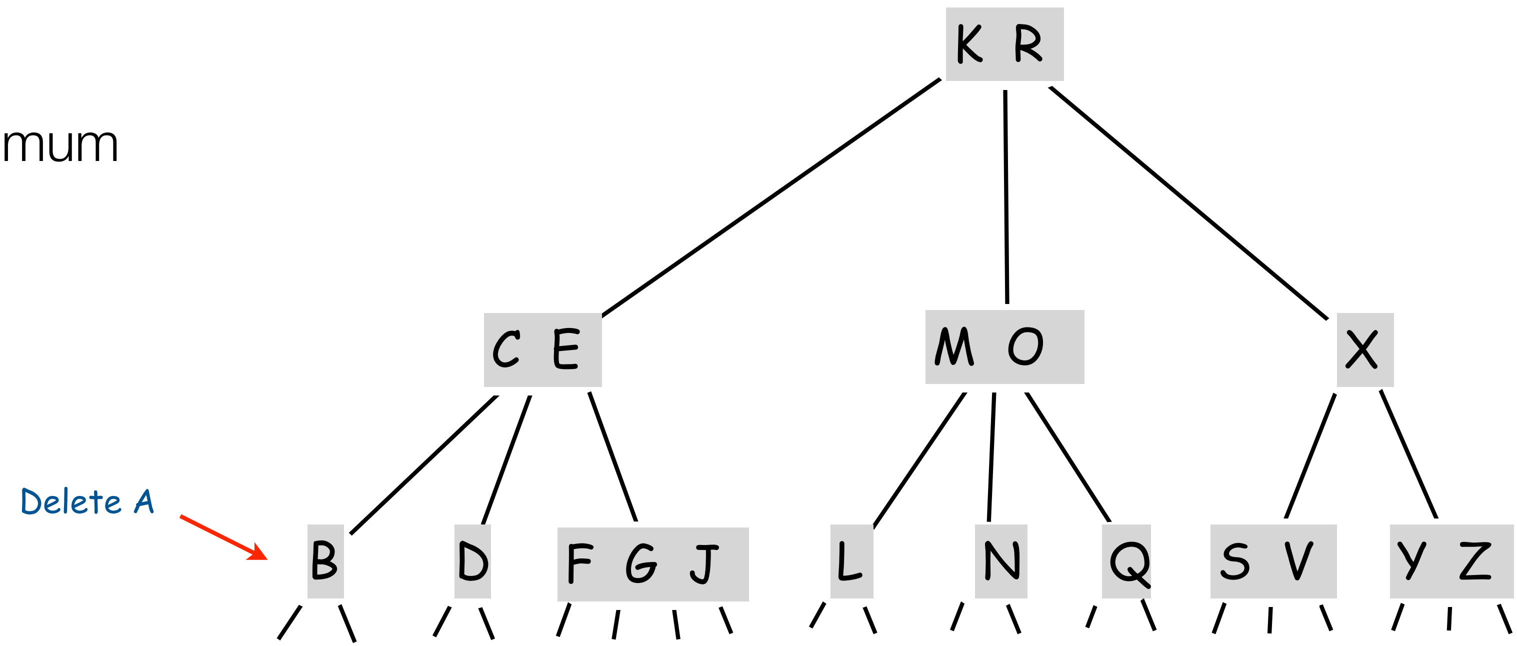
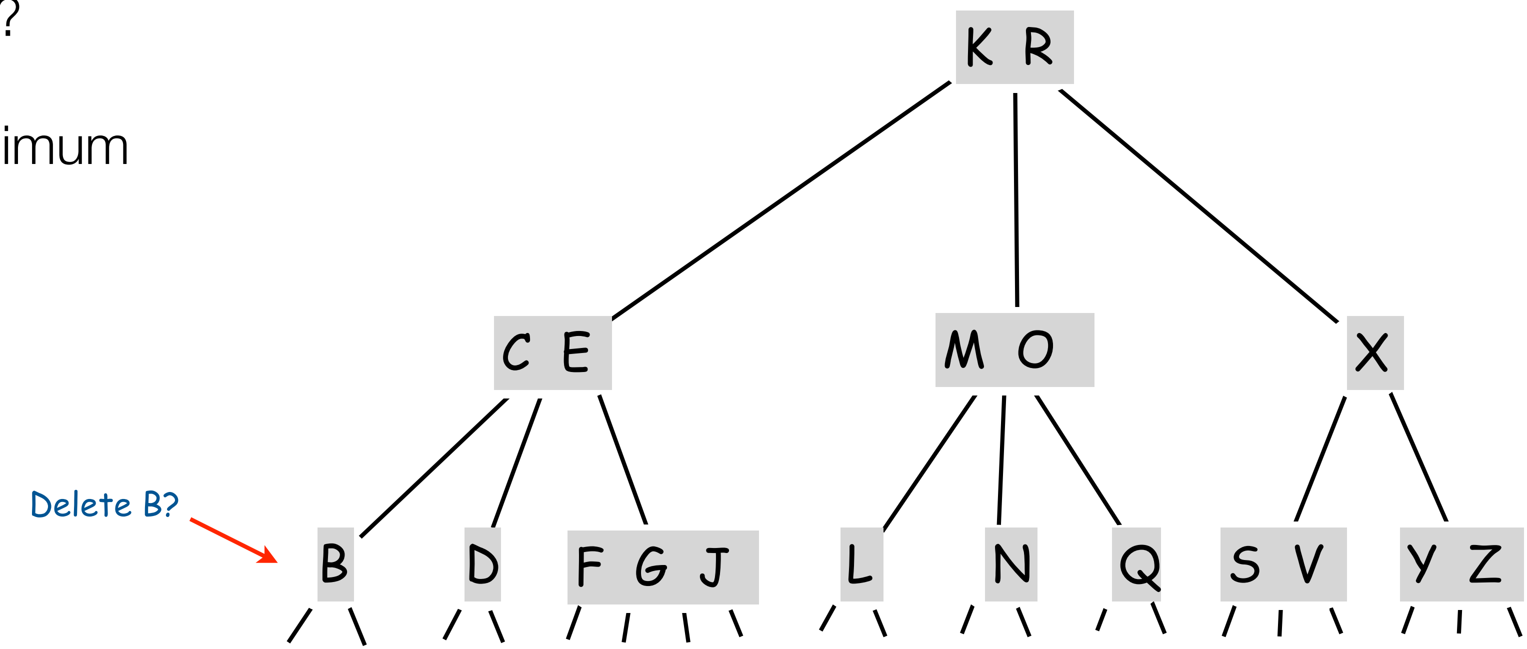


Deletions in 2-3-4 trees

Delete minimum:

- minimum always in leftmost leaf
- If 3- or 4-node: delete key
- 2-node??

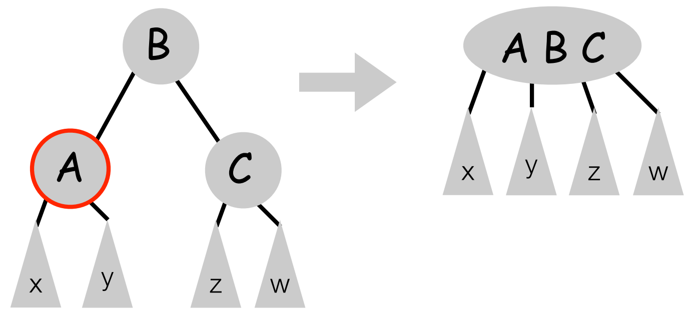
Ex. Delete minimum



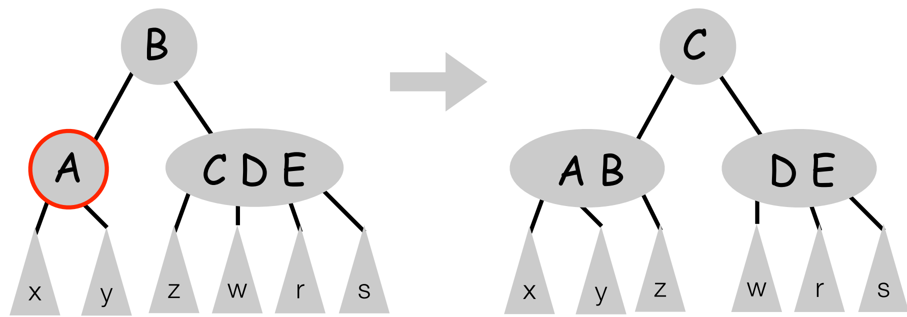
Deletions in 2-3-4 trees

Idea: On the way down maintain the invariant that current node is not a 2-node.

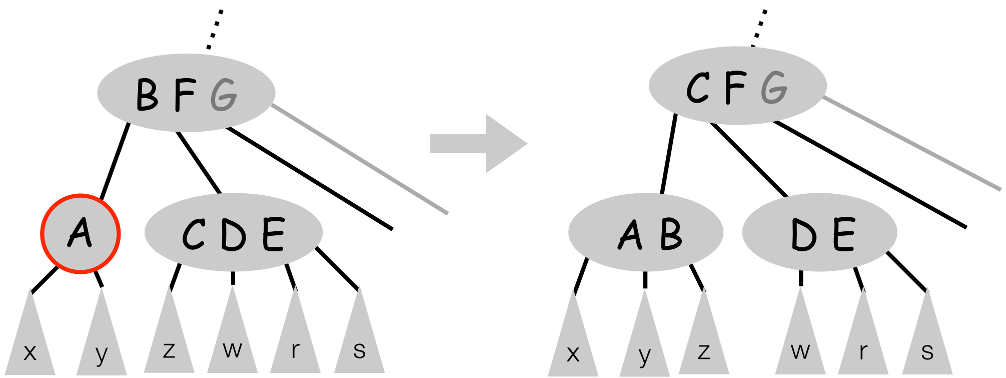
- Child of root and root is a 2-node:



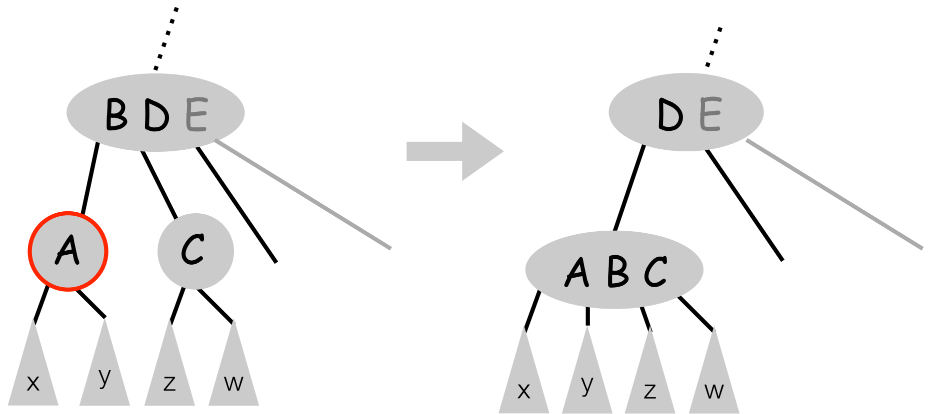
or



- on the way down:



or

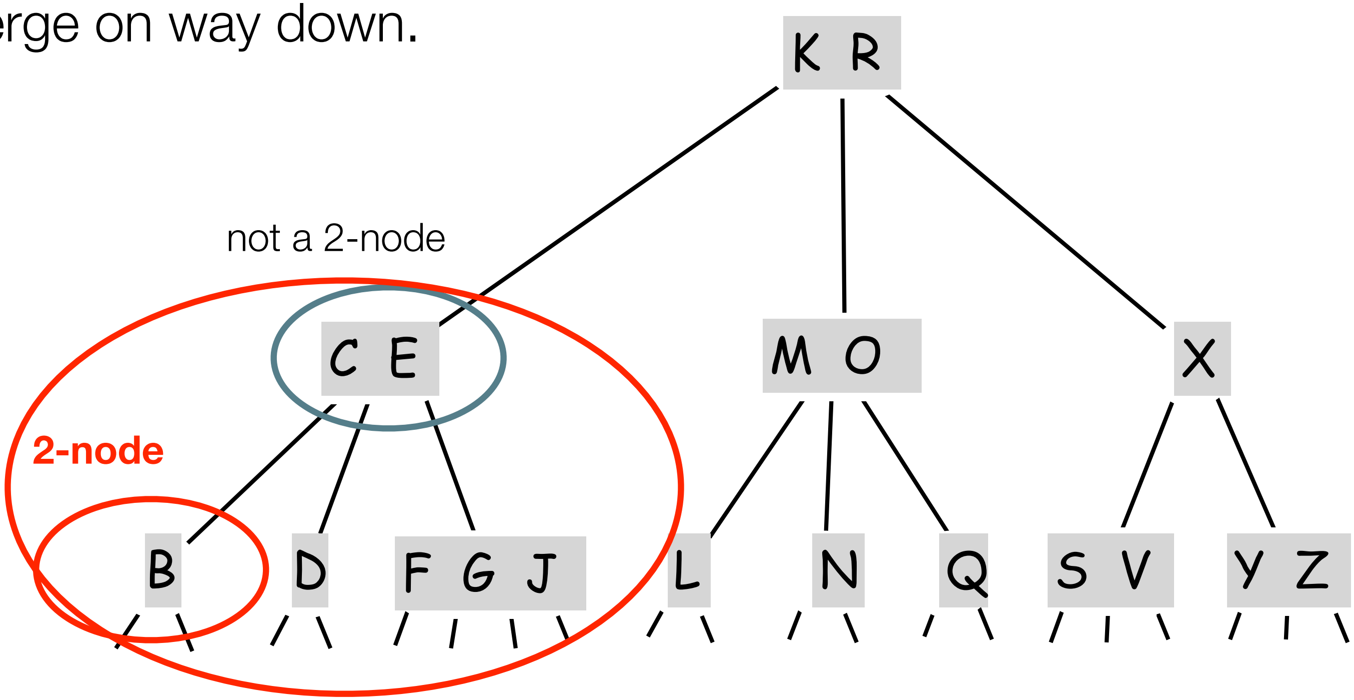


Deletions in 2-3-4 trees

Delete minimum:

- minimum always in leftmost leaf
- If 3- or 4-node: delete key
- 2-node: split/merge on way down.

Ex. Delete minimum



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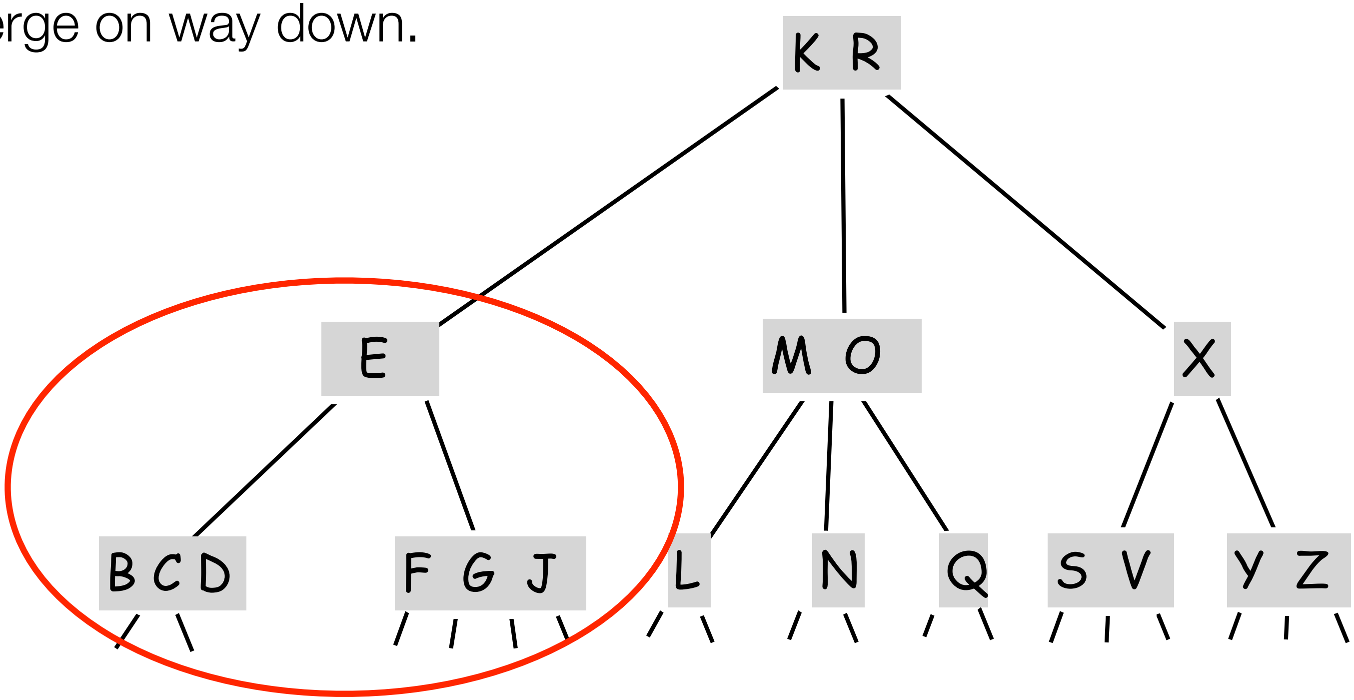

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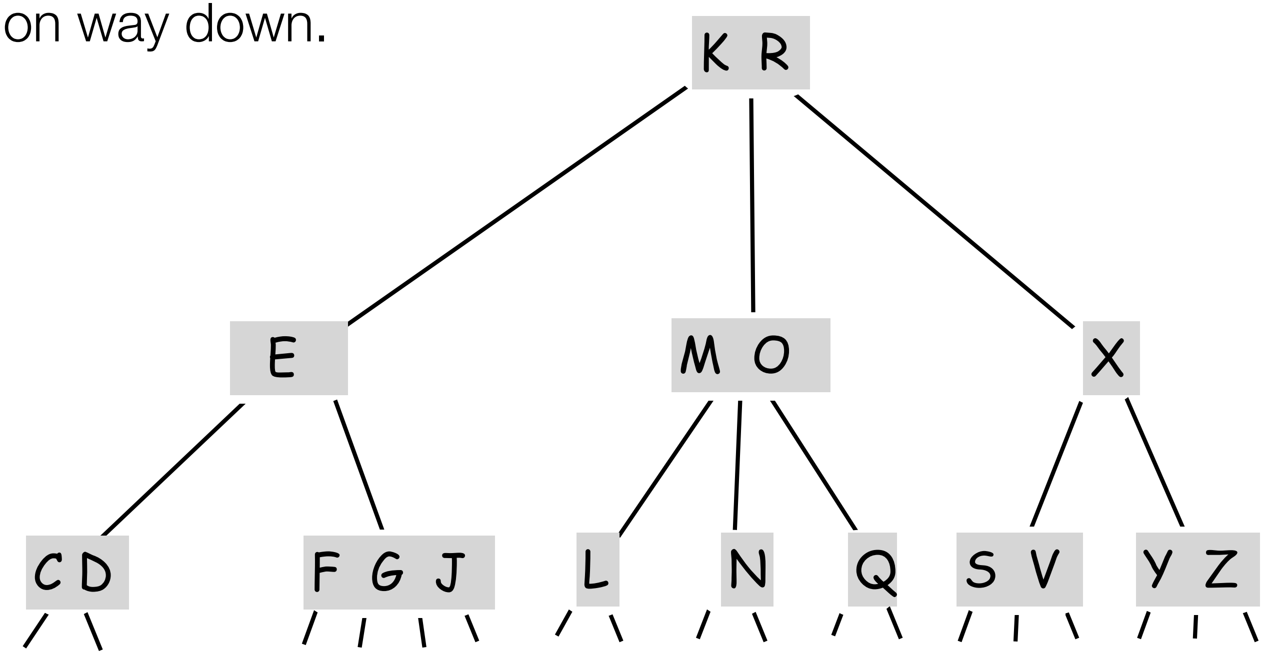
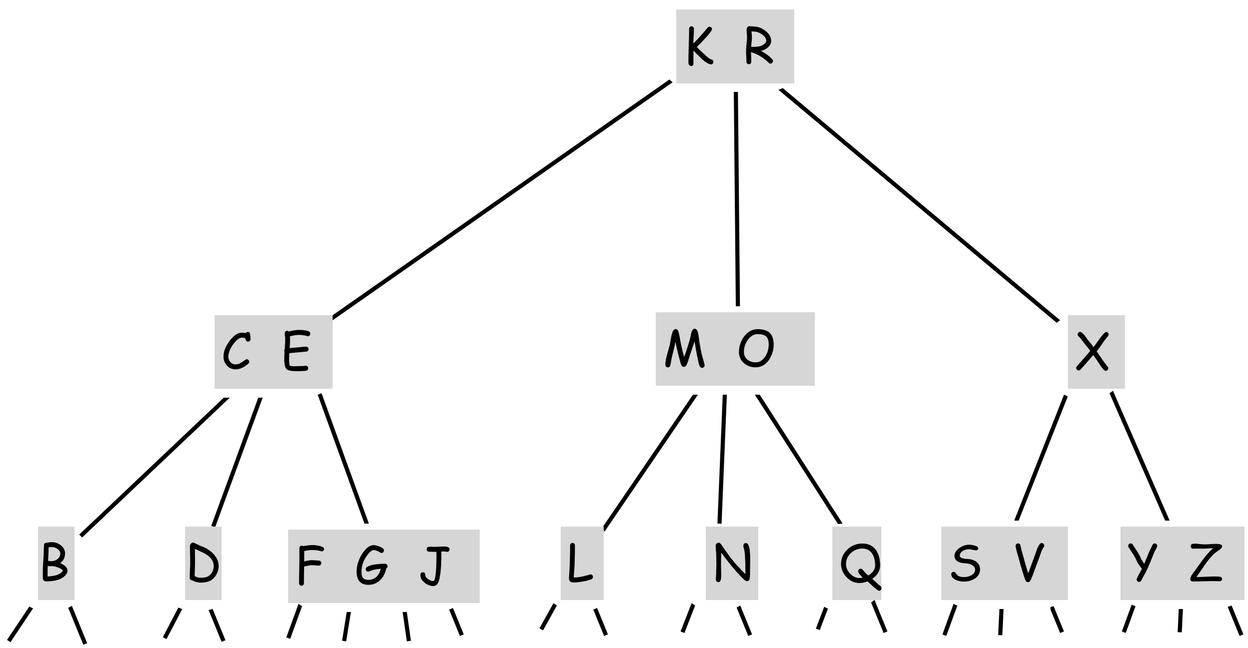


Deletions in 2-3-4 trees

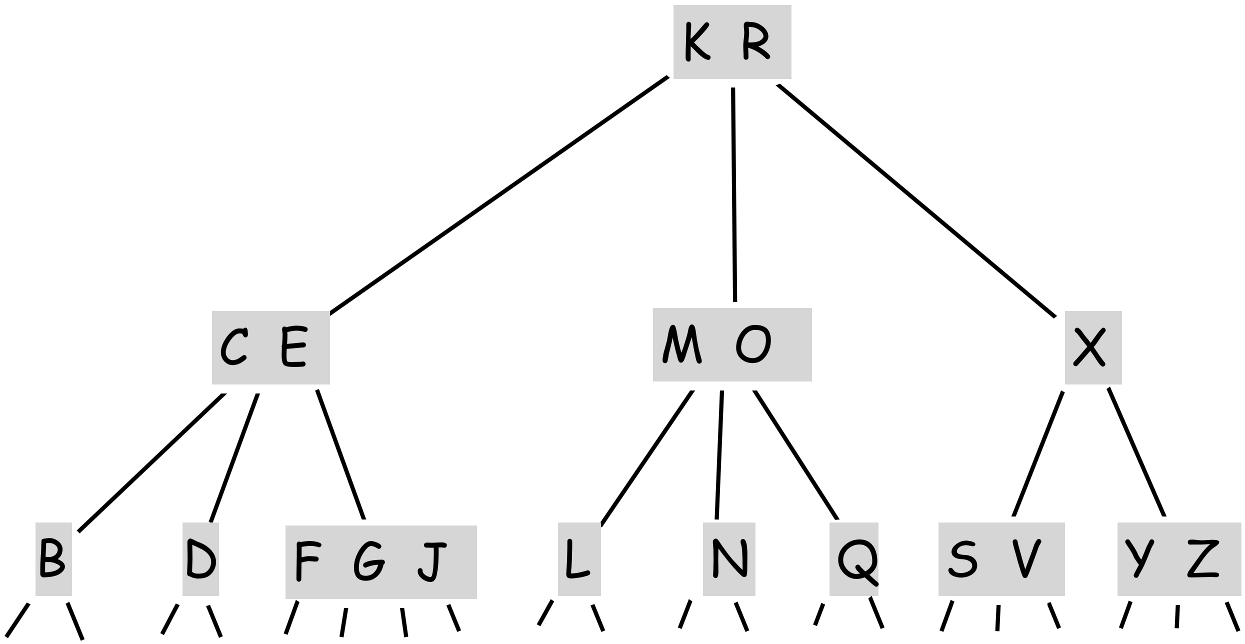
Delete:



Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node



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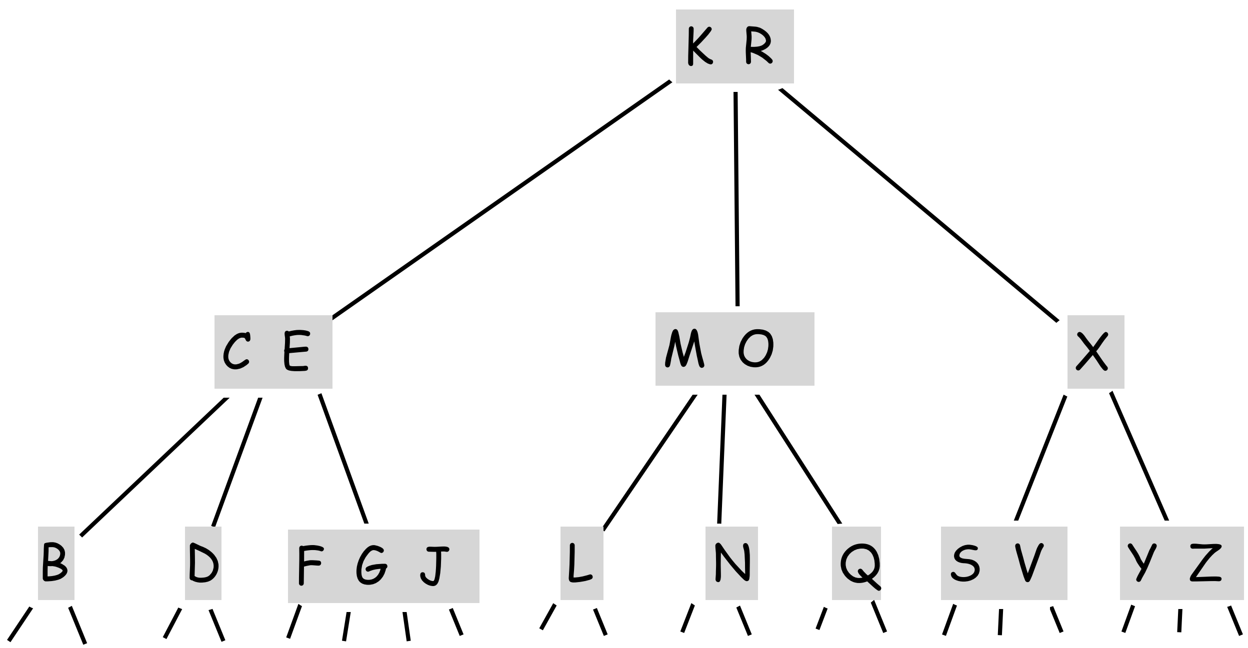

Deletions in 2-3-4 trees

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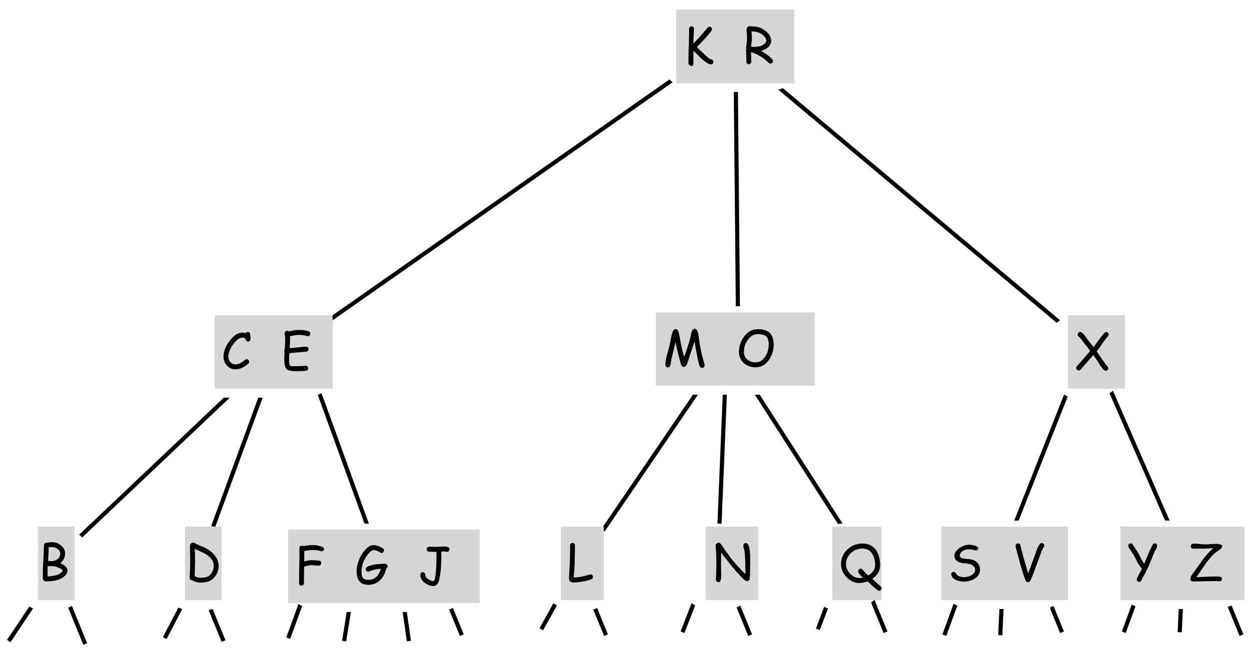

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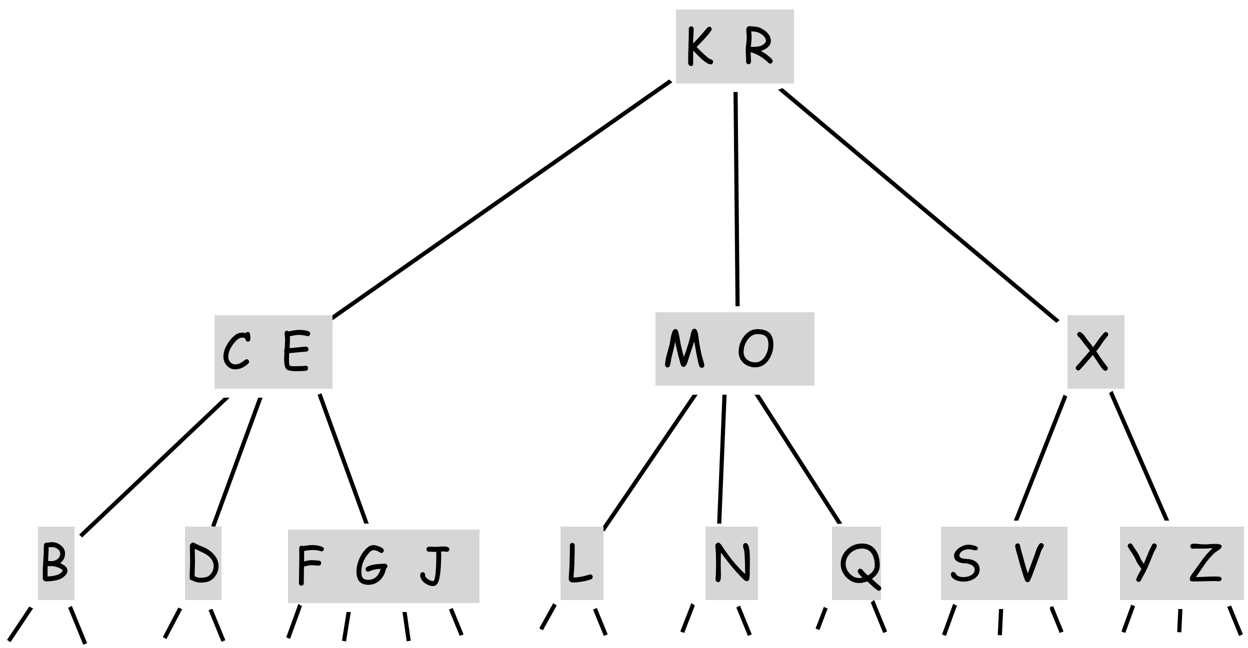
Ex. Delete K



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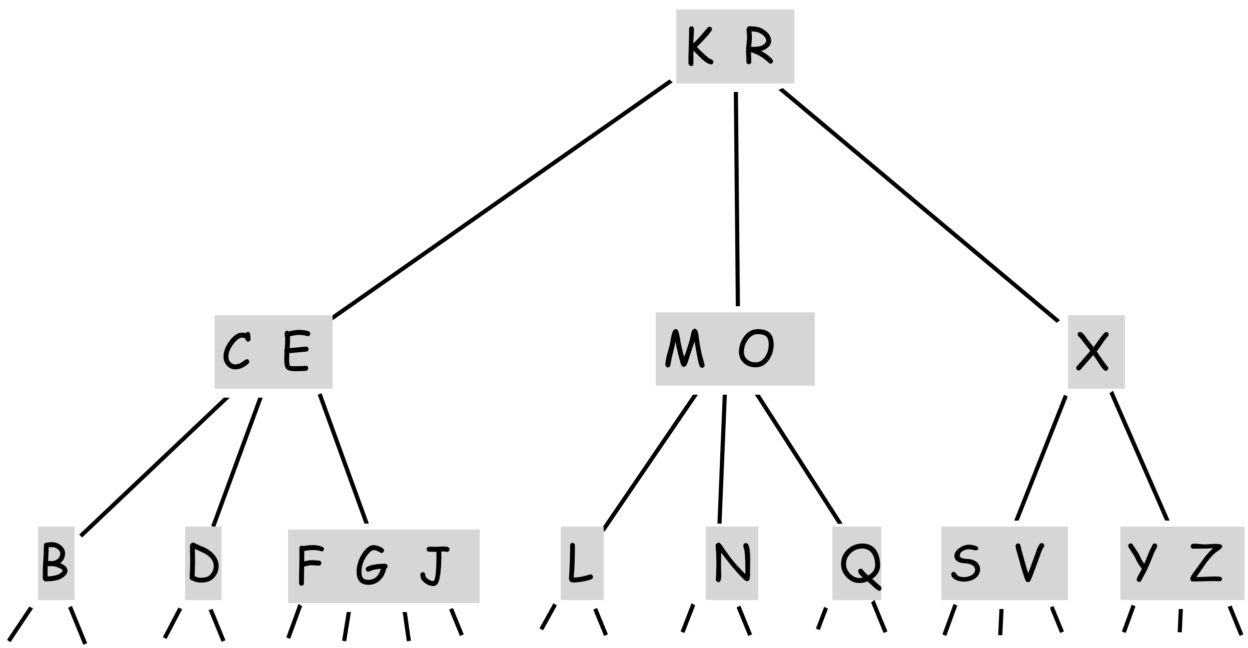

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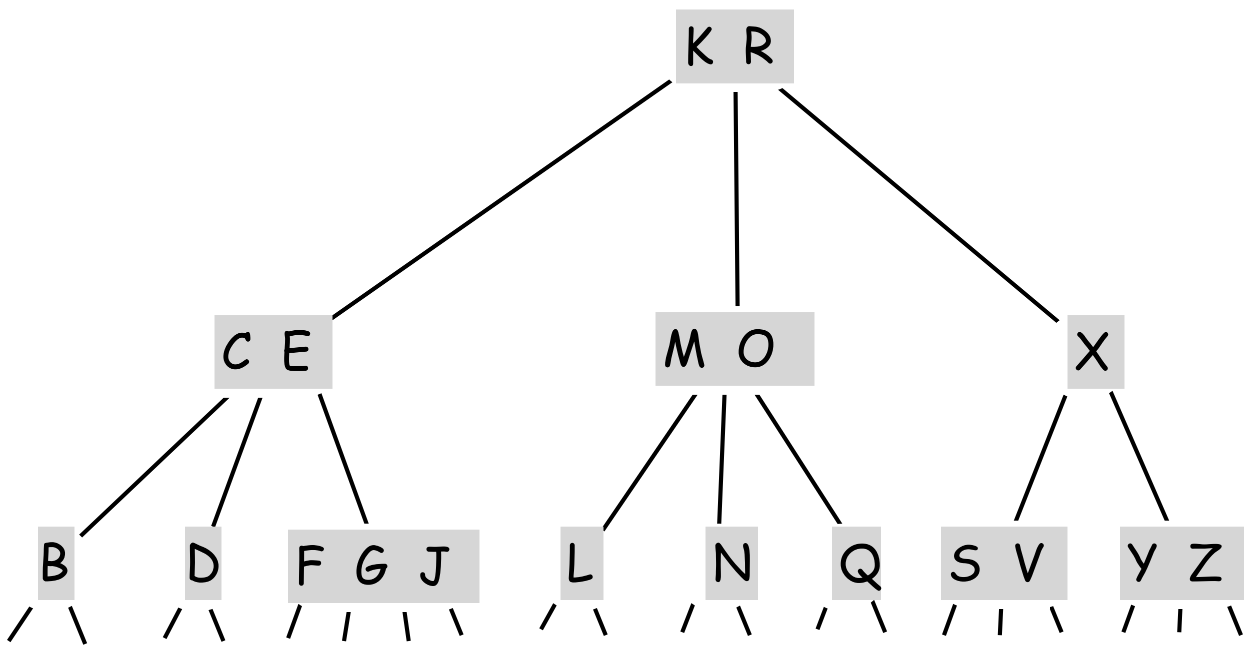

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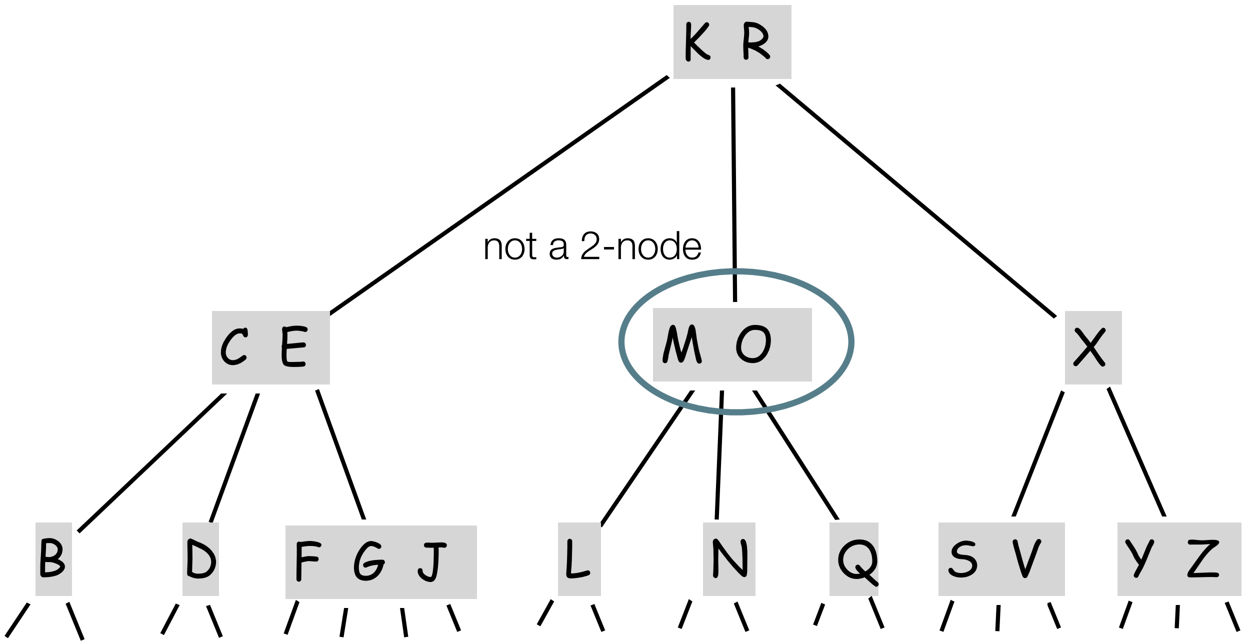

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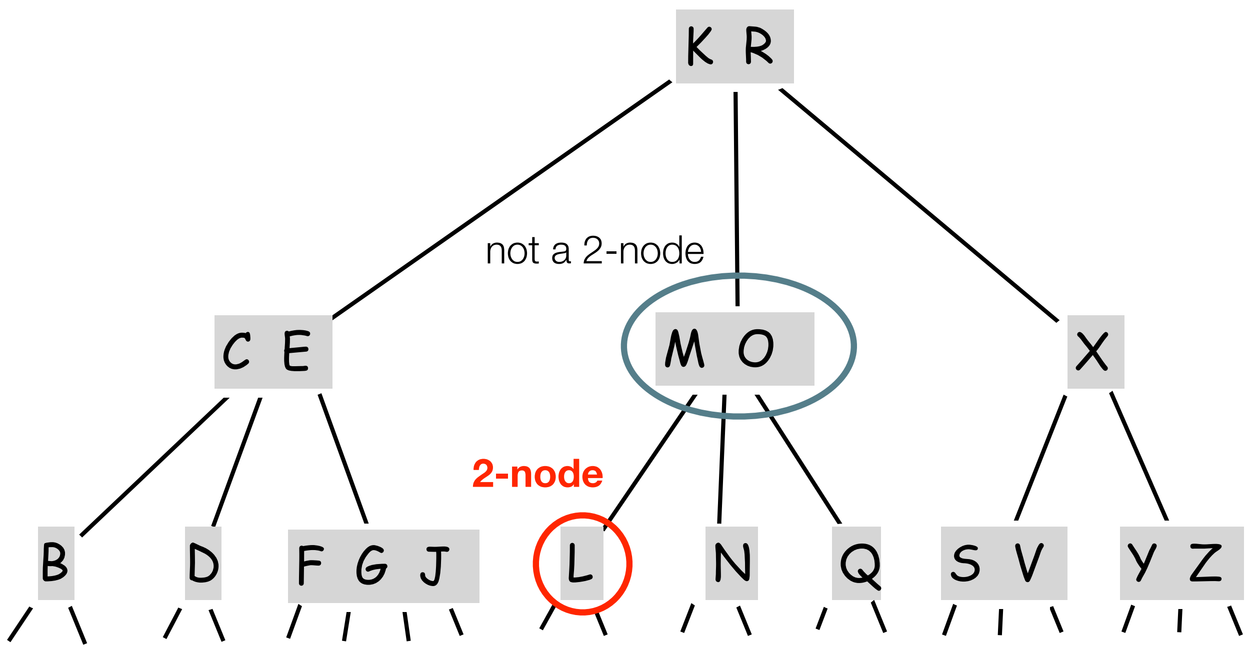

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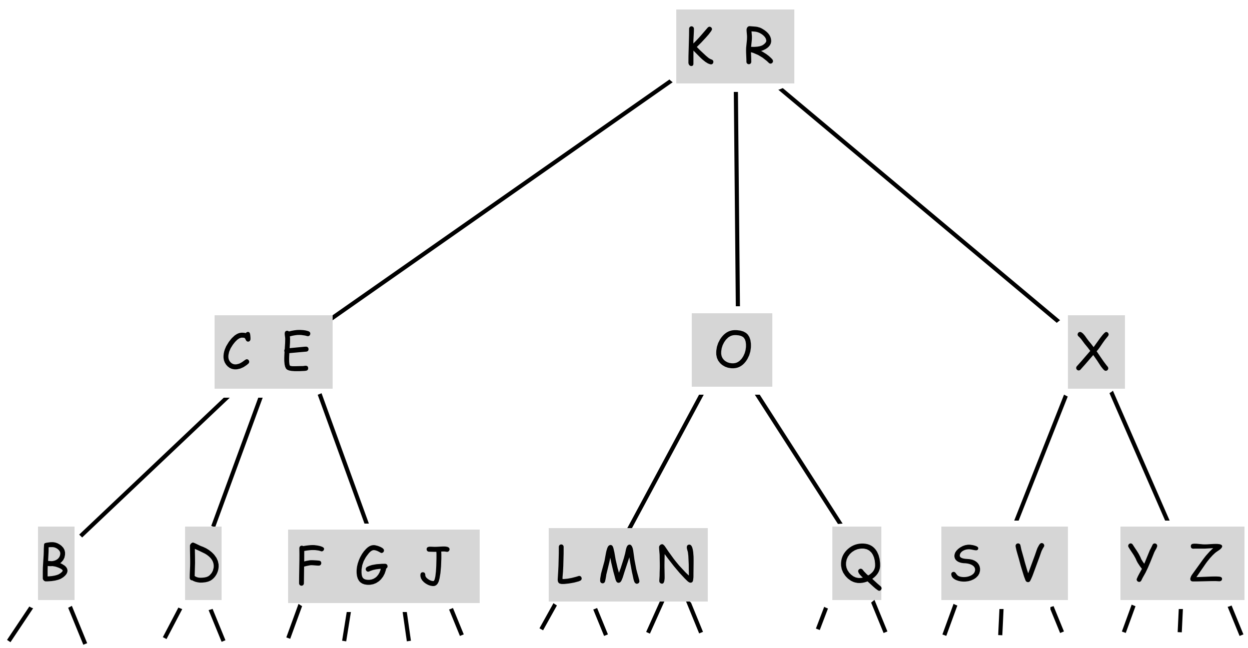

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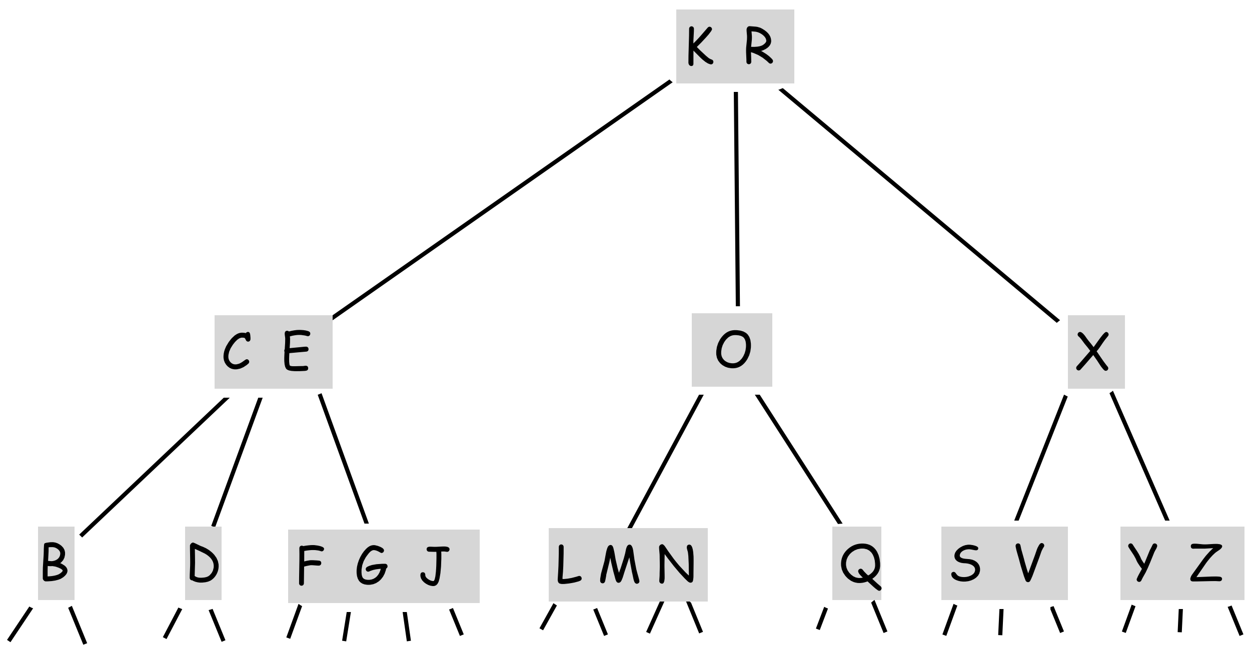
Ex. Delete K

- Find successor
- Delete L from leaf



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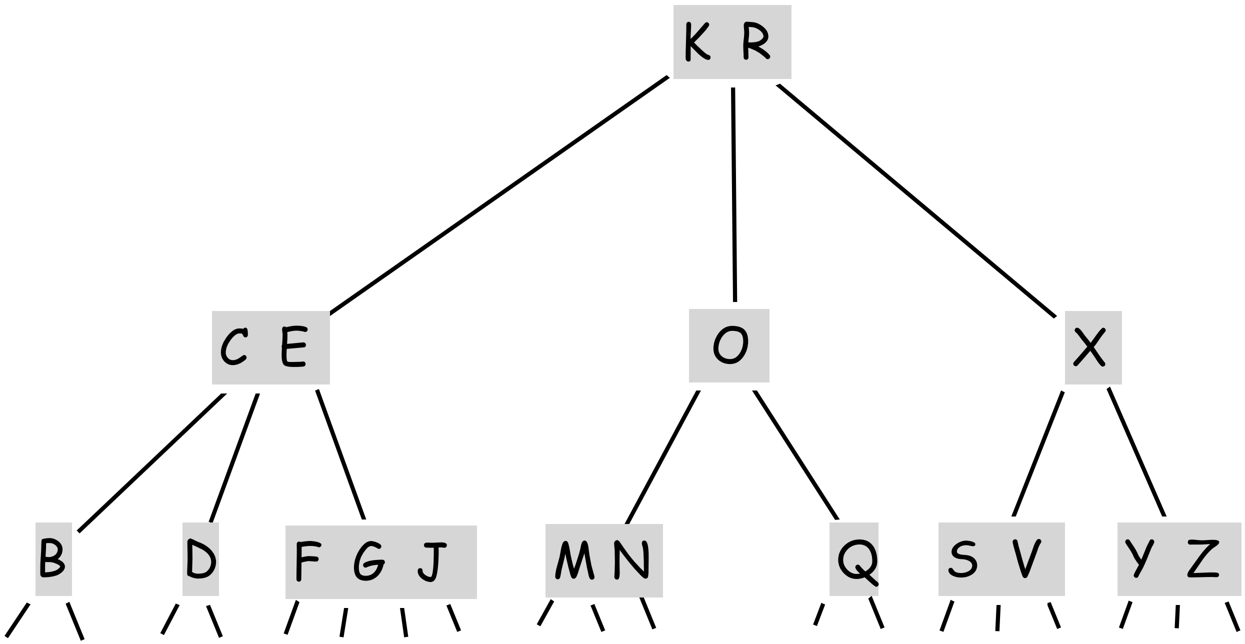

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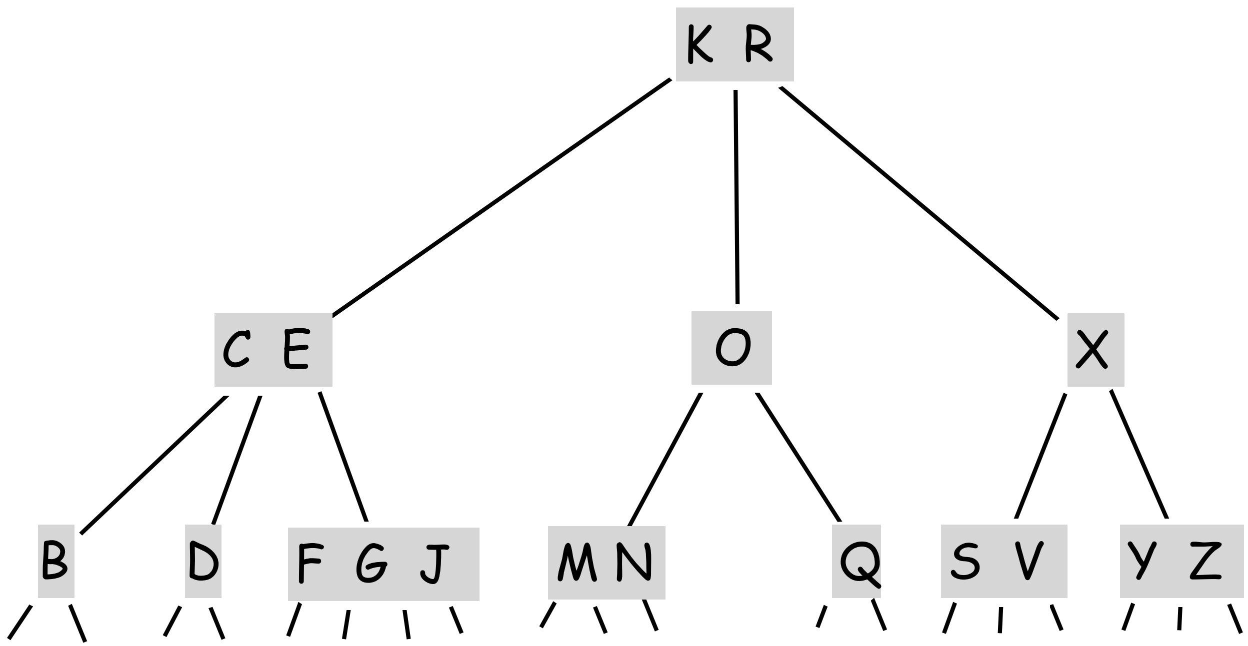
Ex. Delete K

- Find successor
- Delete L from leaf
- Replace K with L



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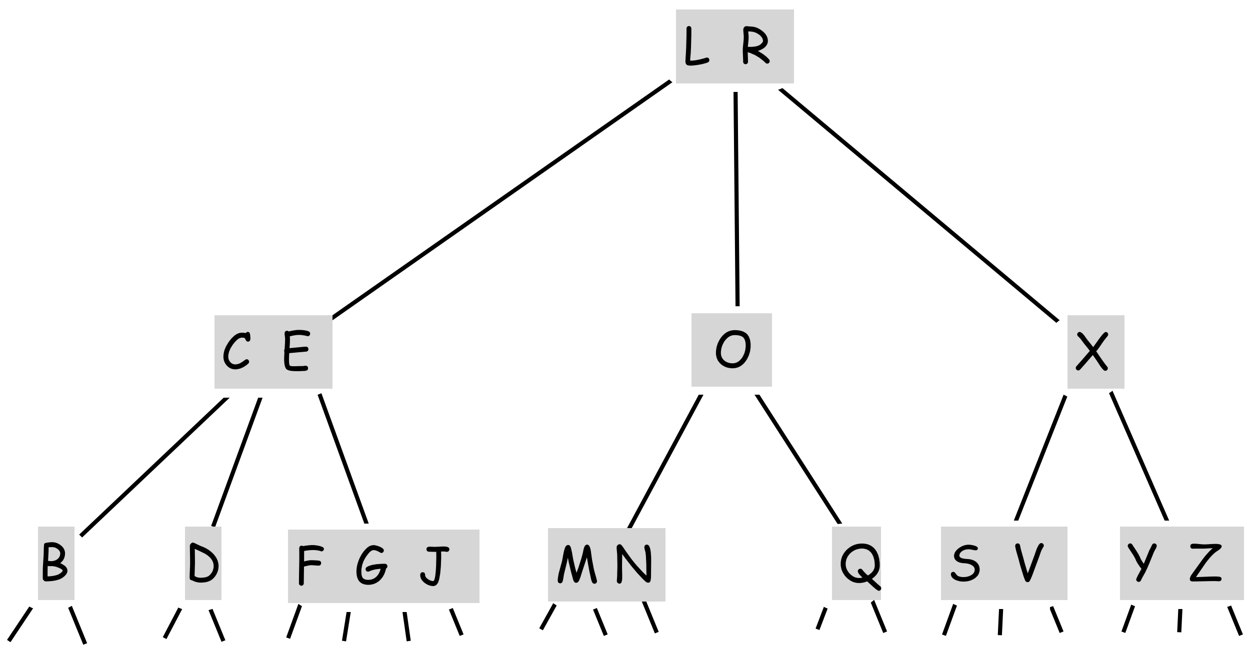
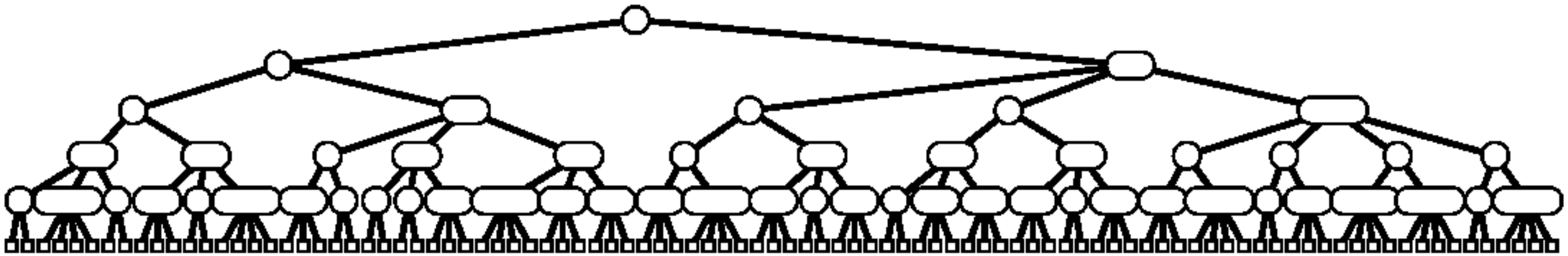


2-3-4 Tree: Balance

Property. All paths from root to leaf have same length.



Tree height.

Worst case: lg N [all 2-nodes]

Best case: $\log_4 N = 1/2 \lg N$ [all 4-nodes]

Between 10 and 20 for a million nodes.

Between 15 and 30 for a billion nodes.

Dynamic set implementations

Worst case running times

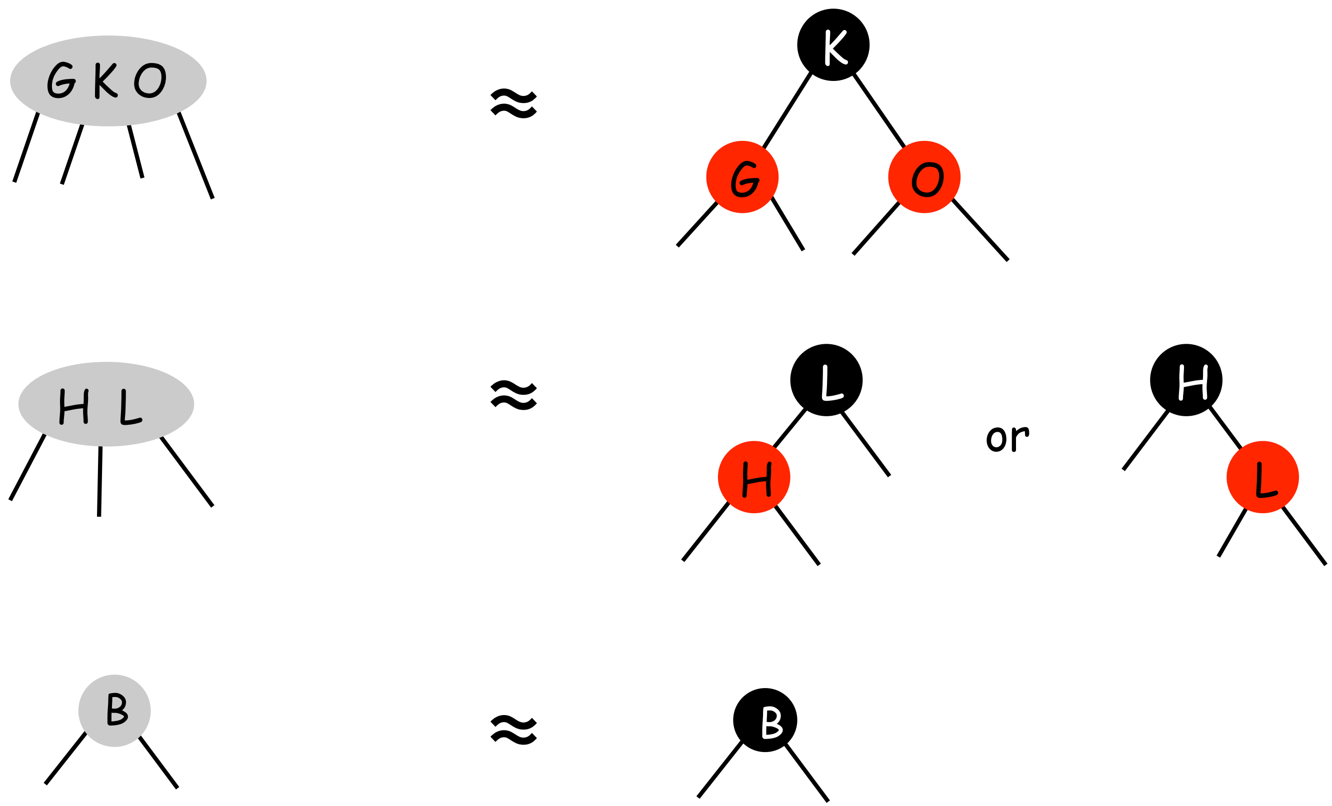
Implementation	search	insert	delete	minimum	maximum	successor	predecessor
linked lists	O(n)	O(1)	O(1)	O(n)	O(n)	O(n)	O(n)
ordered array	O(log n)	O(n)	O(n)	O(1)	O(1)	O(log n)	O(log n)
BST	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)
2-3-4 tree	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)

Red-black trees

Red-black tree (Guibas-Sedgewick, 1979)

Represent 2-3-4 tree as a binary search tree

- Use colors on nodes to represent 3- and 4-nodes.



Red-black tree (Guibas-Sedgewick, 1979)

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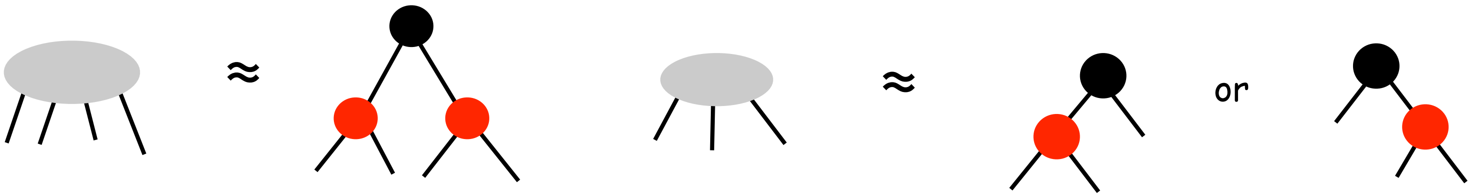
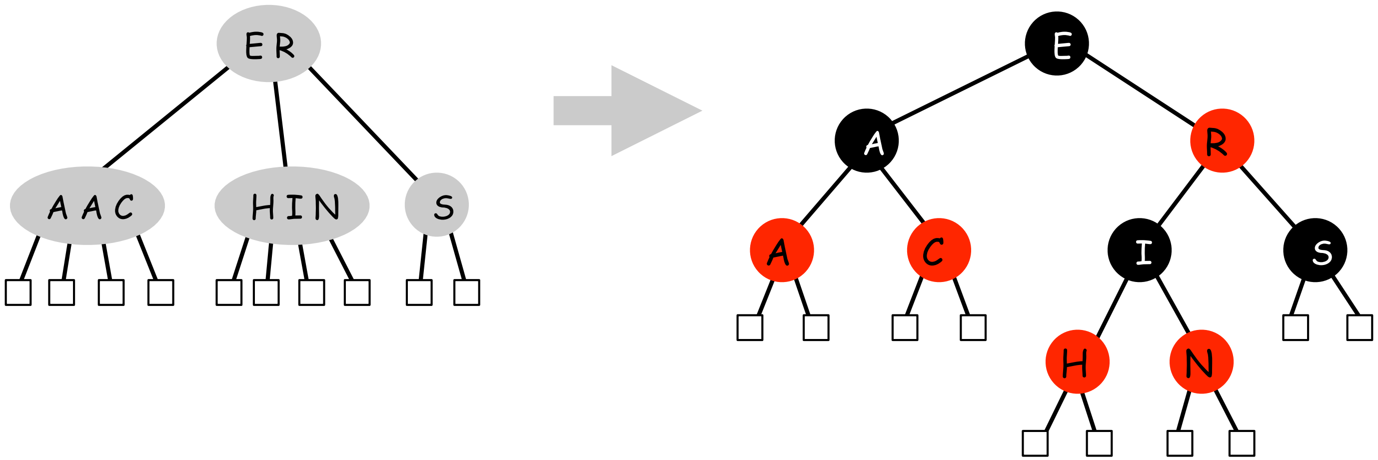


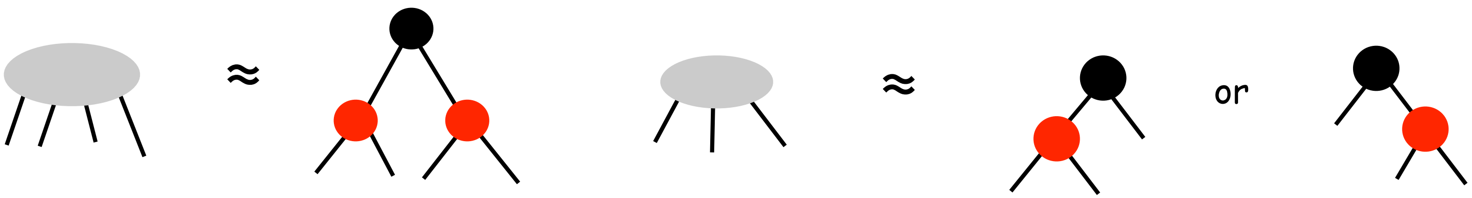
- Connection between 2-3-4 trees and red-black trees:



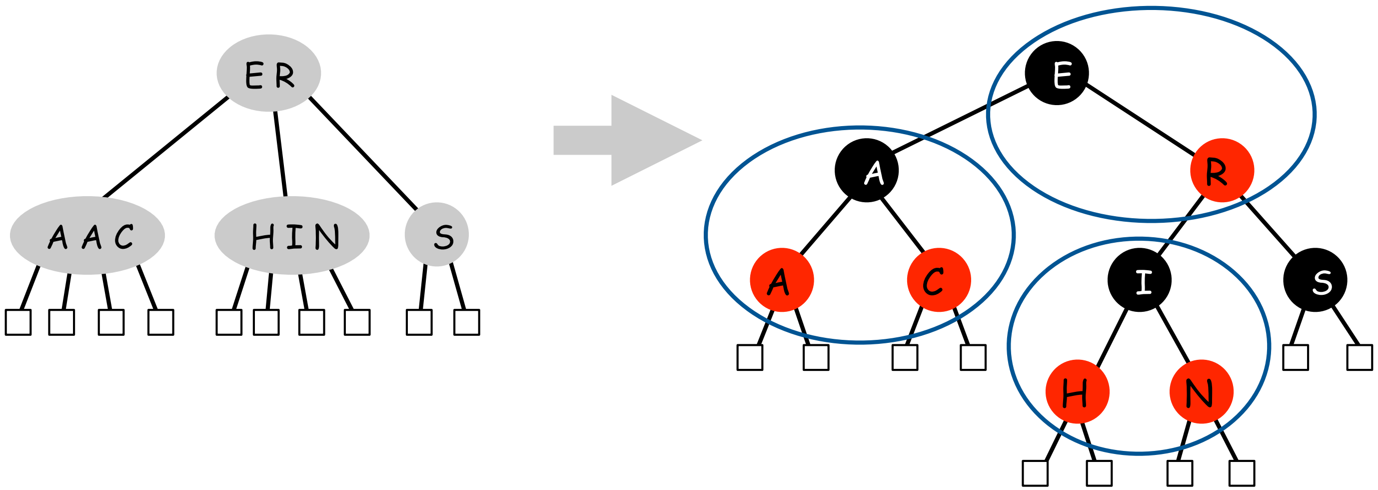
Red-black tree (Guibas-Sedgewick, 1979)

Represent 2-3-4 tree as a binary search tree

- Use colors on nodes to represent 3- and 4-nodes.



- Connection between 2-3-4 trees and red-black trees:



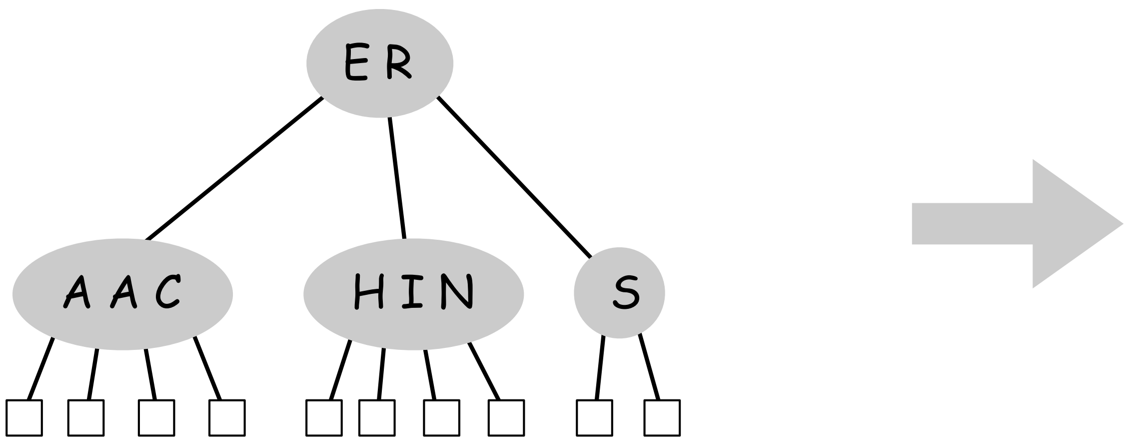
Red-black tree

Properties of red-black trees:

- The root is always black
- All root-to-leaf paths have the same number of black nodes.
- Red nodes do not have red children

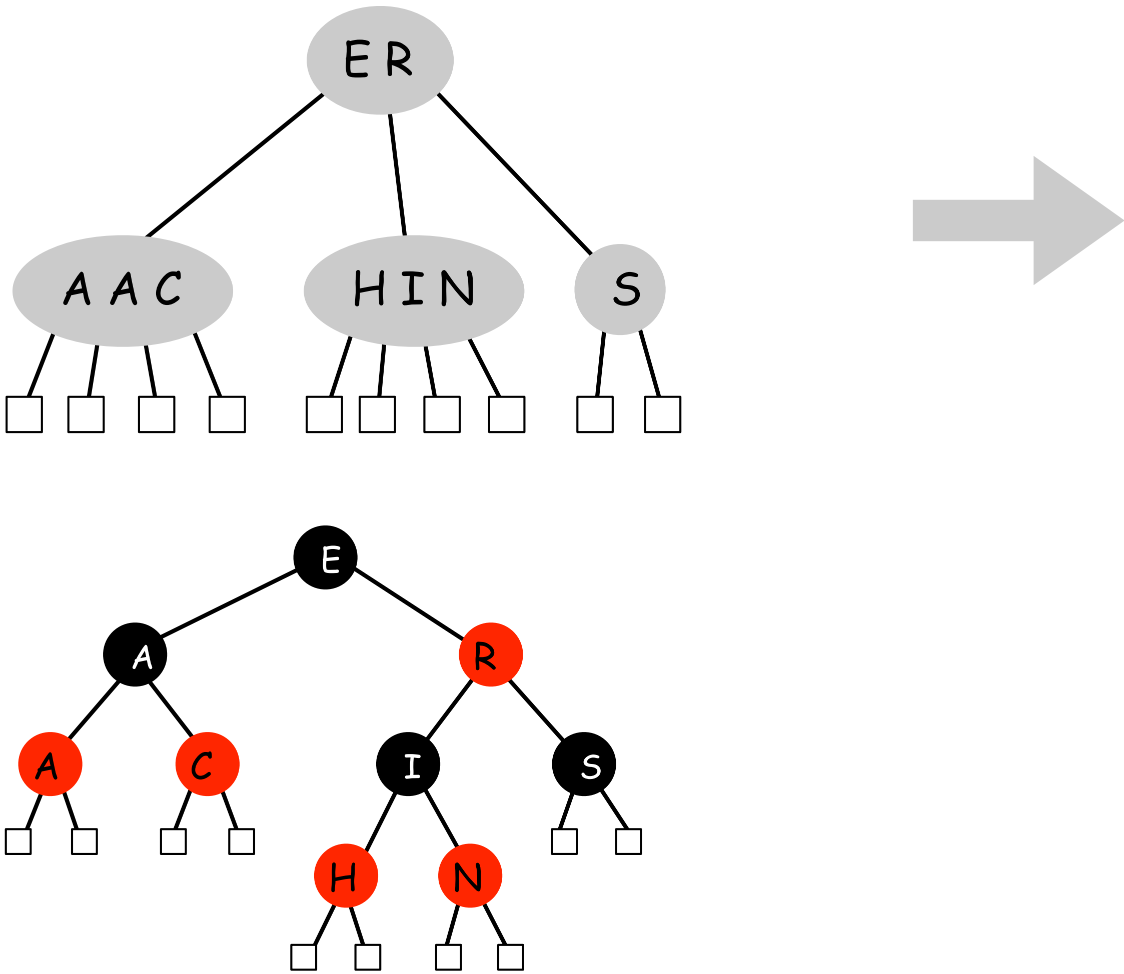
Red-black tree

Connection between 2-3-4 trees and red-black trees:



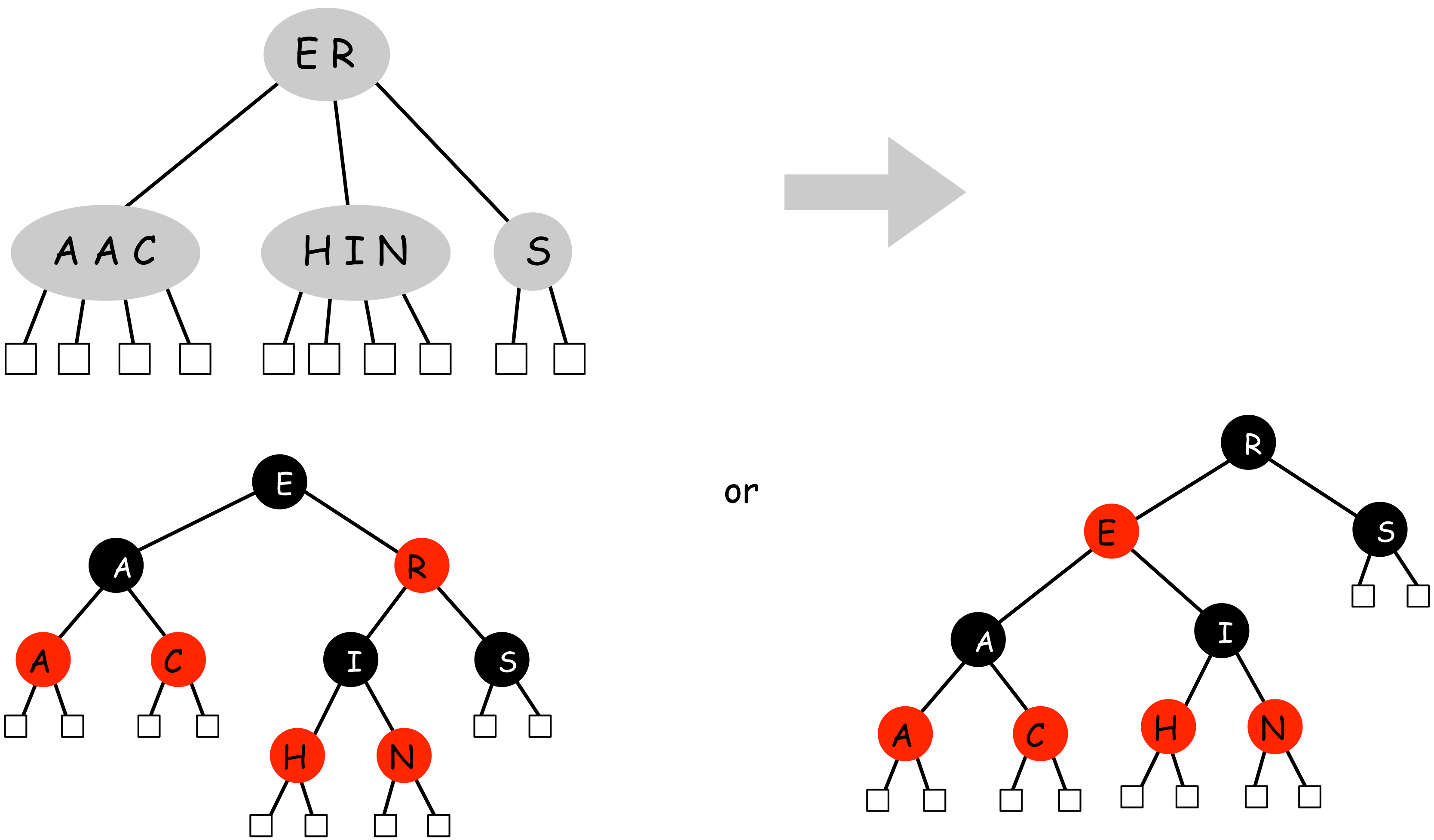
Red-black tree

Connection between 2-3-4 trees and red-black trees:



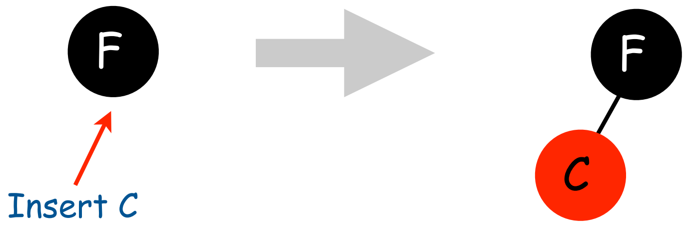
Red-black tree

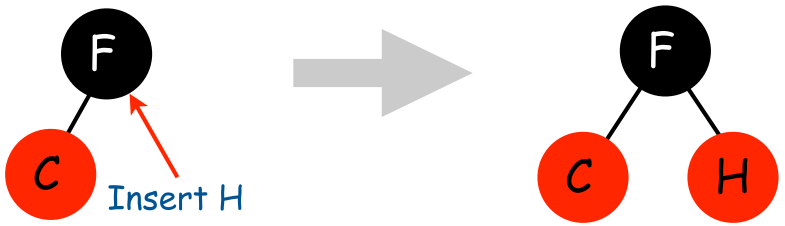
Connection between 2-3-4 trees and red-black trees:



Insertion in red-black trees

Insertion: Insert a new red leaf.

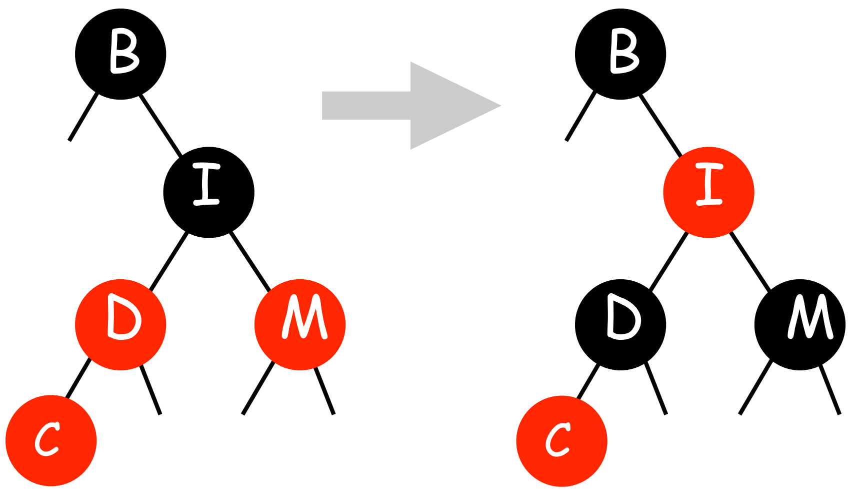




Red-black tree: Parent is red

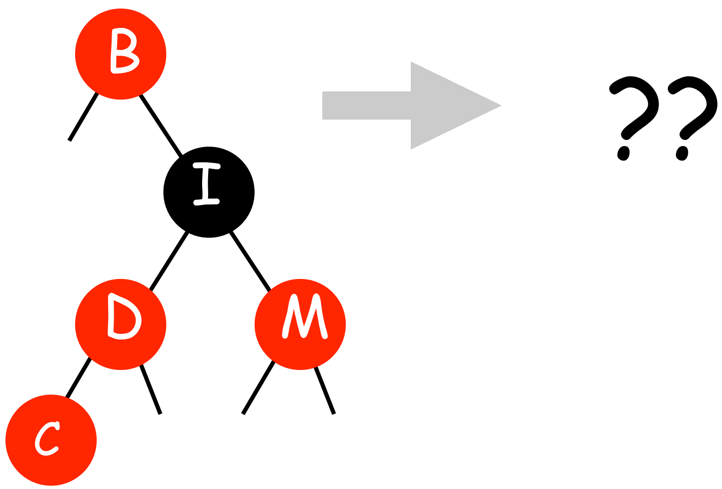
What if the parent is also red?

Easy case:



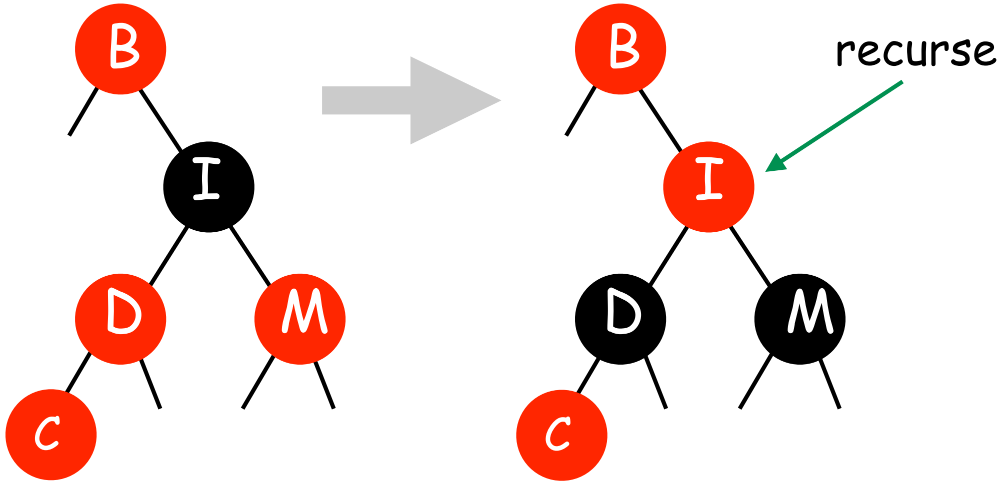
Red-black tree: Parent is red

What if both the parent and the grandparent are red?



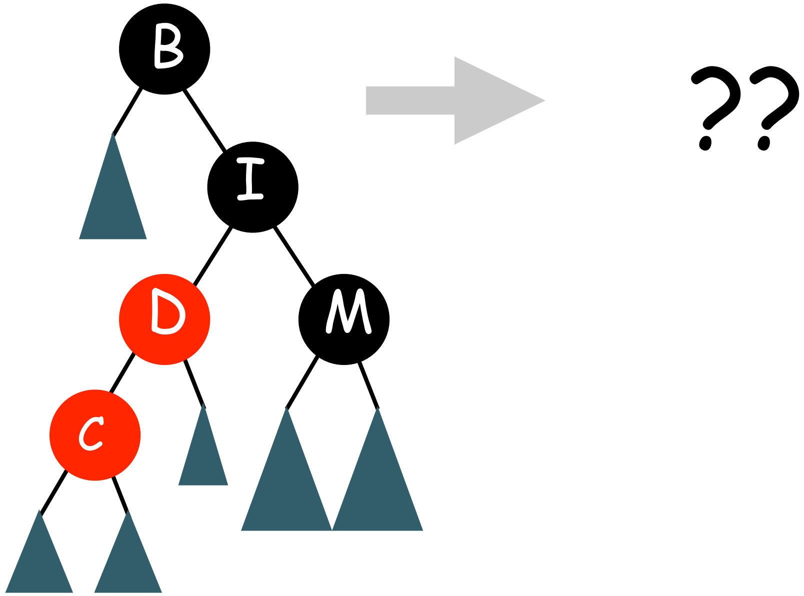
Red-black tree: Parent is red

What if both the parent and the grandparent are red?



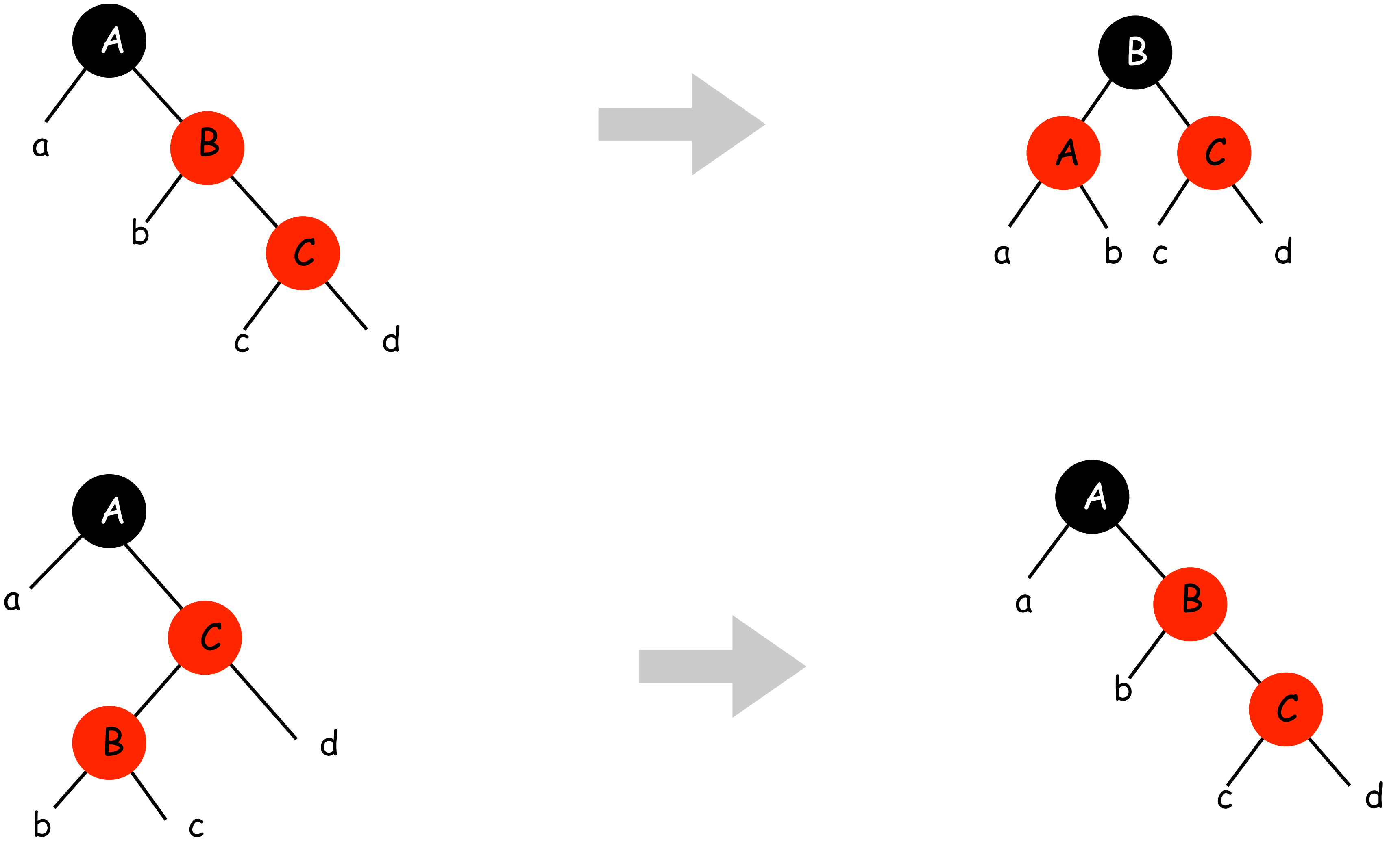
Red-black tree: Parent is red

What if the parent is also red?



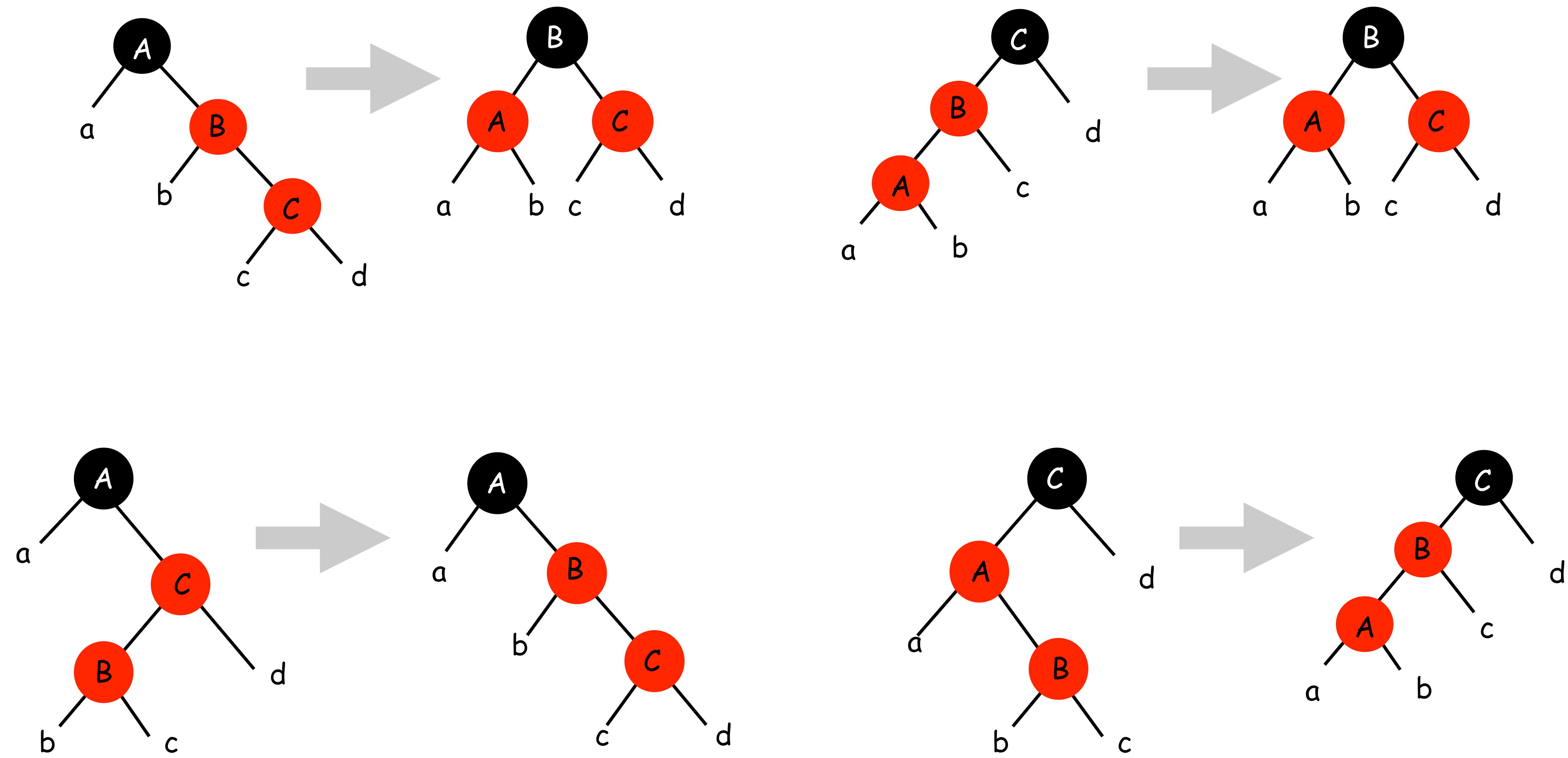
Rotations in red-black trees

Two types of rotations



Rotations in red-black trees

Two types of rotations:



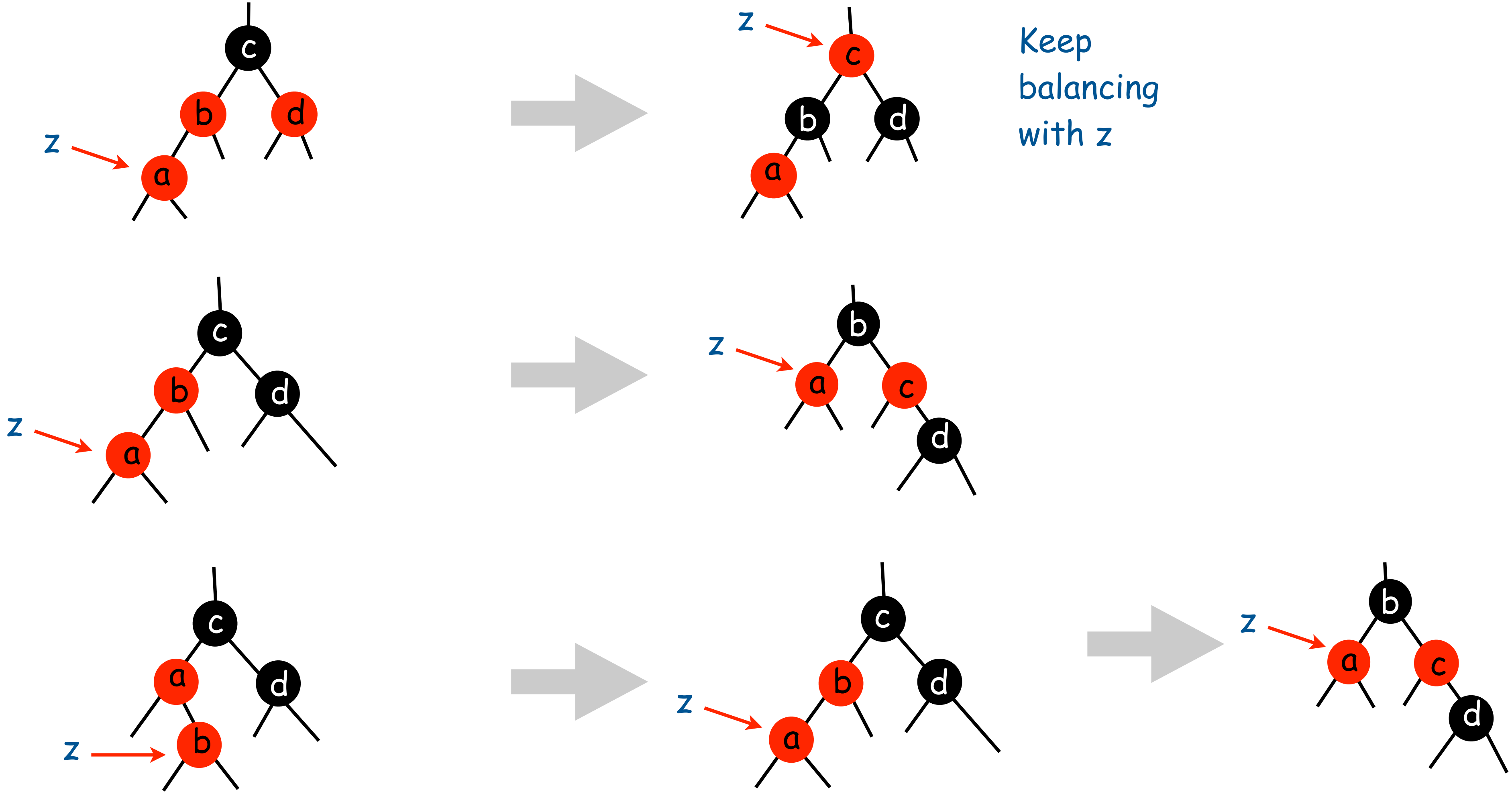
Insertion in red-black tree

Insert x:

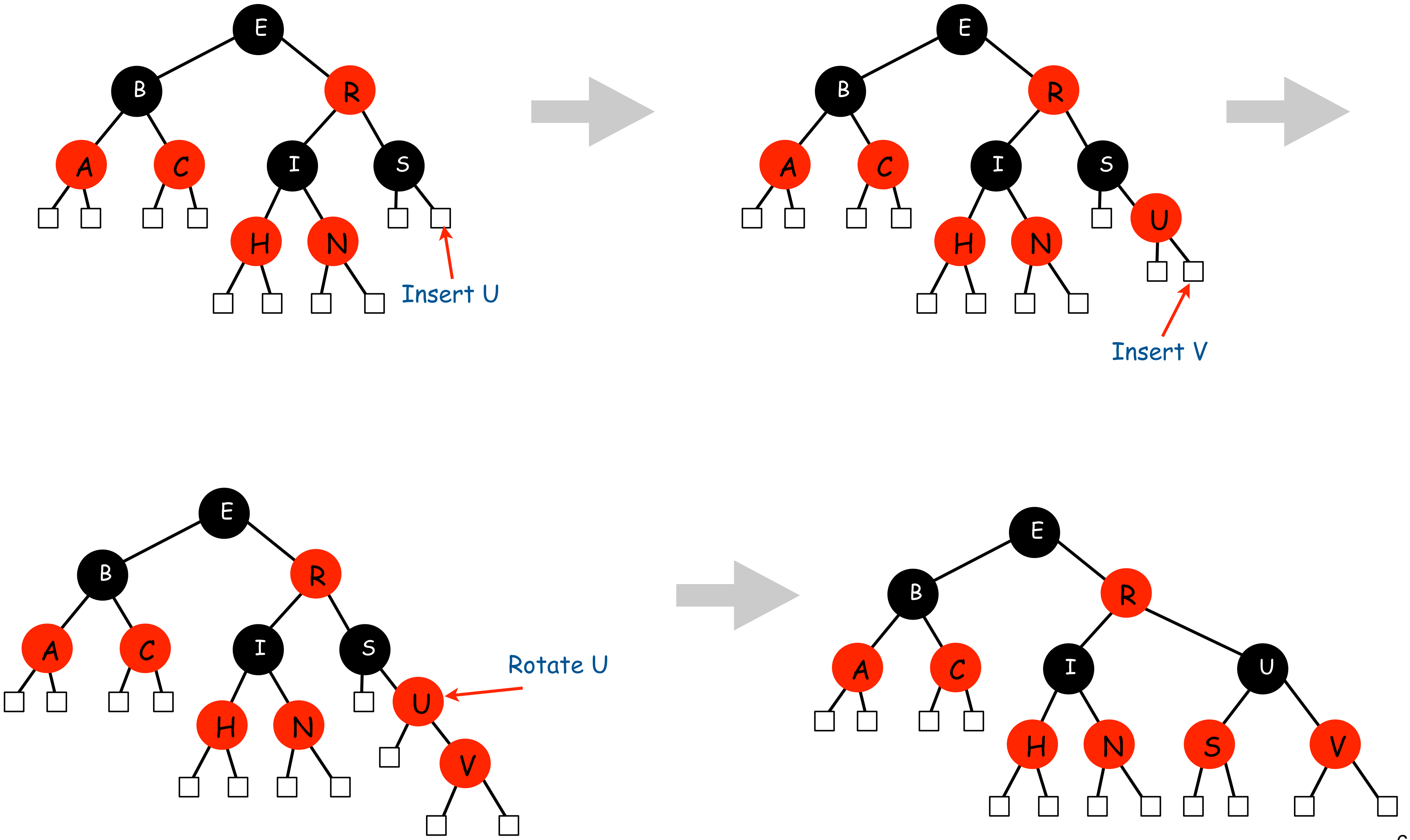
Search to bottom after key (x)

Insert red leaf

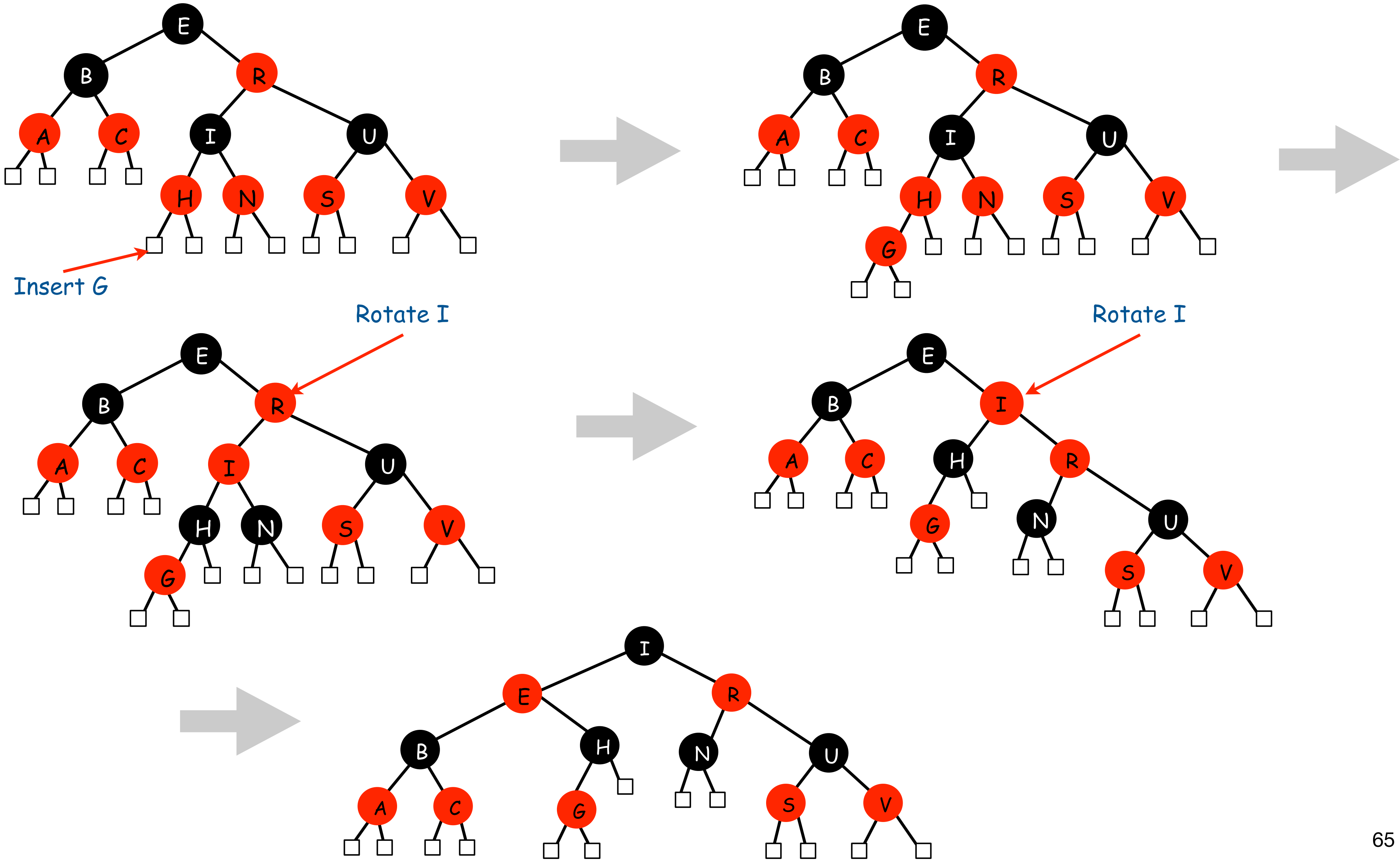
Balance: 3 cases (+ symmetric)



Example



Example



Running times in red-black trees

- Time for insertion:
    - Search to bottom after key: O(h)
    - Insert red leaf: O(1)
    - Perform recoloring and rotations on way up: O(h)
        - Can recolor many times (but at most h)
        - At most 2 rotations.
- Total O(h).

- Time for search
    - Same as BST: O(h)

- Height: O(log n)

Dynamic set implementations

Worst case running times

Implementation	search	insert	delete	minimum	maximum	successor	predecessor
linked lists	O(n)	O(1)	O(1)	O(n)	O(n)	O(n)	O(n)
ordered array	O(log n)	O(n)	O(n)	O(1)	O(1)	O(log n)	O(log n)
BST	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)
2-3-4 tree	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)
red-black tree	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)

Balanced trees: implementations

Redblack trees:

Java: `java.util.TreeMap, java.util.TreeSet.`

C++ STL: map, multimap, multiset.

Linux kernel: `linux/rbtree.h.`