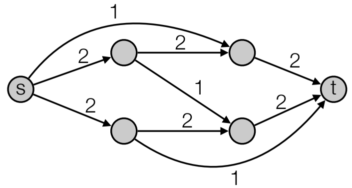
Network Flow II

Inge Li Gørtz

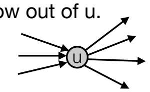
KT 7.3, 7.5, 7.6

Network Flow



- Network flow:
  - graph G=(V,E).
  - Special vertices s (source) and t (sink).
  - Every edge e has a capacity c(e) ≥ 0.
  - Flow:
    - **capacity constraint:** every edge e has a flow 0 ≤ f(e) ≤ c(e).
    - **flow conservation:** for all u ≠ s, t: flow into u equals flow out of u.

$$\sum_{v:(v,u)\in E} f(v,u) = \sum_{v:(u,v)\in E} f(u,v)$$



    - Value of flow f is the sum of flows out of s minus sum of flows into s:

$$v(f) = \sum_{v:(s,v)\in E} f(e) - \sum_{v:(v,s)\in E} f(e) = f^{out}(s) - f^{in}(s)$$

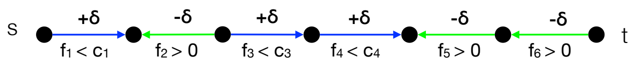
  - **Maximum flow problem:** find s-t flow of maximum value

Today

- Applications
- Finding good augmenting paths. Edmonds-Karp and scaling algorithm.

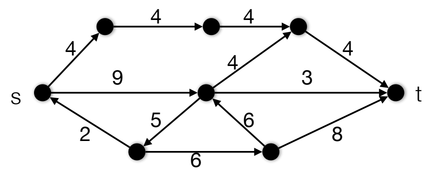
Ford-Fulkerson

- Find (any) augmenting path and use it.
- Augmenting path (definition different than in CLRS): s-t path where
  - forward edges have leftover capacity
  - backwards edges have positive flow



  - Can add extra flow: $\min(c_1 - f_1, f_2, c_3 - f_3, c_4 - f_4, f_5, f_6) = \delta$

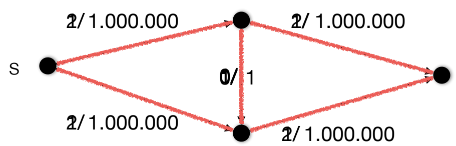
- To find augmenting path use DFS or BFS:



Ford-Fulkerson

- Integral capacities:
  - Each augmenting path increases flow with at least 1.
  - At most $v(f)$ iterations
  - Find augmenting path via DFS/BFS: $O(m)$
  - Total running time: $O(m \cdot v(f))$
- Lemma. If all the capacities are integers, then there is a maximum flow where the flow on every edge is an integer.

- Bad example for Ford-Fulkerson:



Edmonds-Karp

- Find *shortest* augmenting path and use it.
- Augmenting path (definition different than in CLRS): s-t path where
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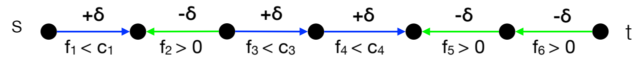

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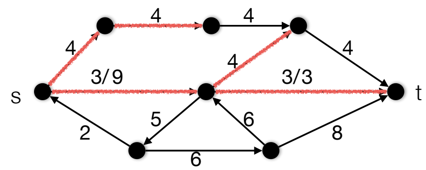
- To find augmenting path use ***BFS***:



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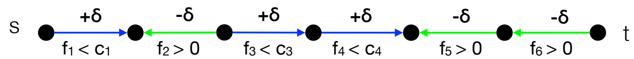

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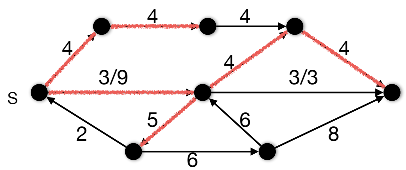

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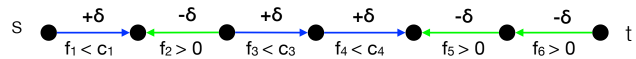

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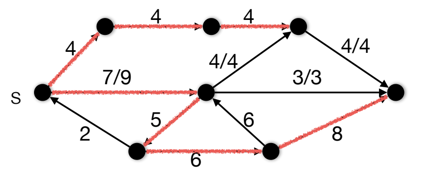

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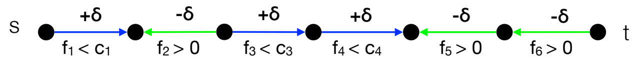

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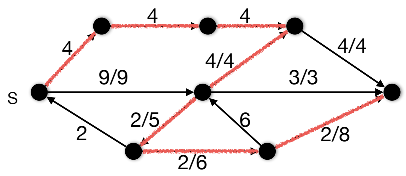

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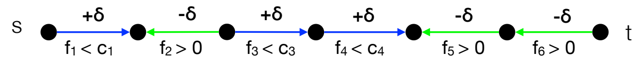

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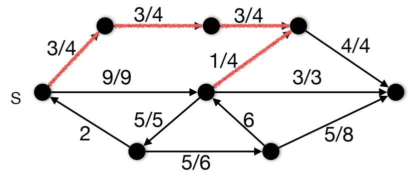
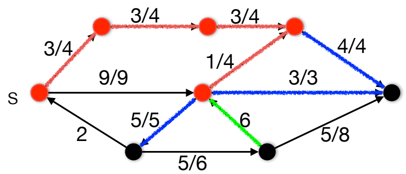


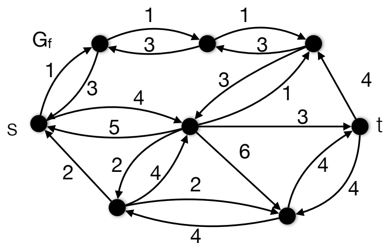
Find a minimum cut

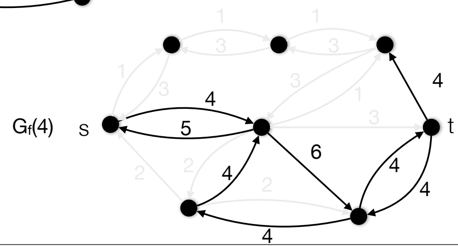
- When there are no more augmenting s-t paths:
- Find all augmenting paths from s.
- The nodes S that can be reached by these augmenting paths form the left side of a minimum cut.
  - edges out of S have $c_e = f_e$.
  - edges into S have $f_e = 0$.
  - Capacity of the cut equals the flow.



Scaling algorithm

- Scaling parameter Δ
- Only consider edges with capacity at least Δ in residual graph $G_f(\Delta)$.
- Example: Δ = 4





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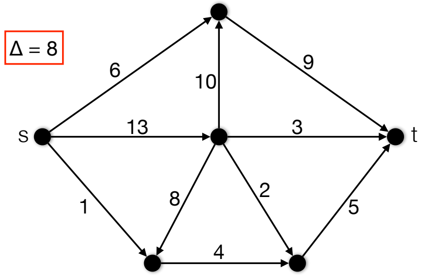

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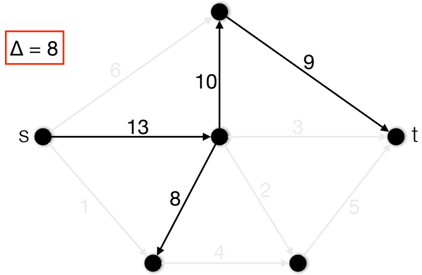

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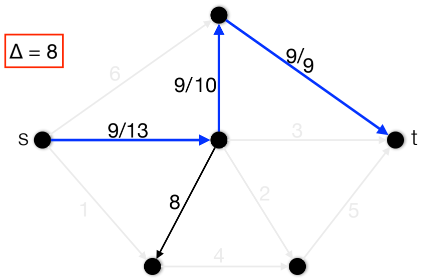

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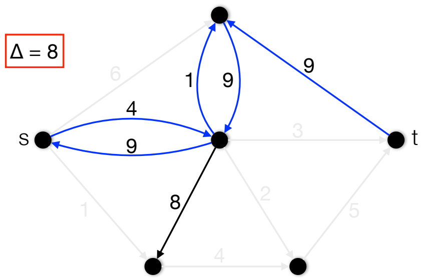

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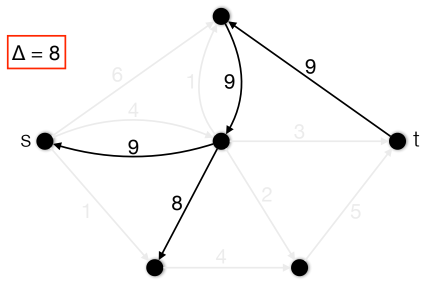

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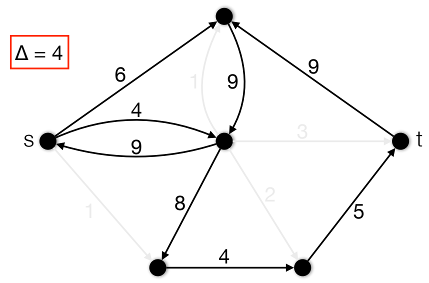

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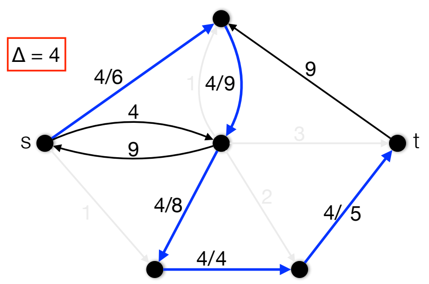

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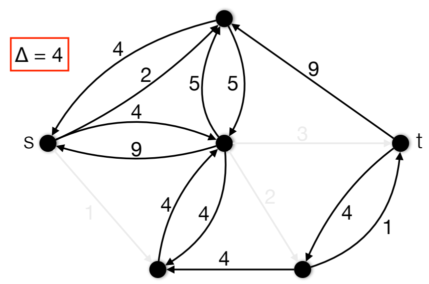

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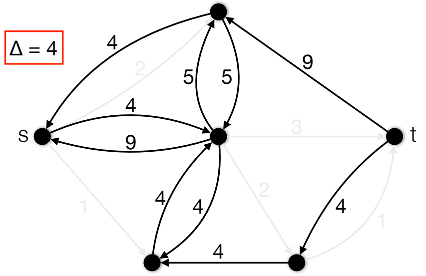

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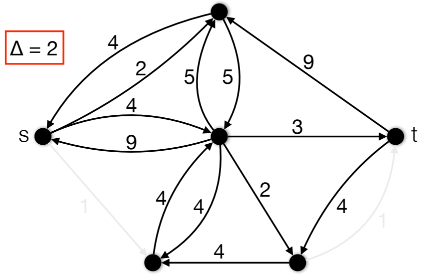

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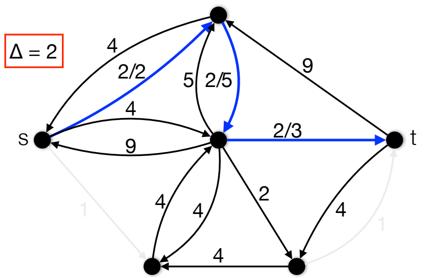

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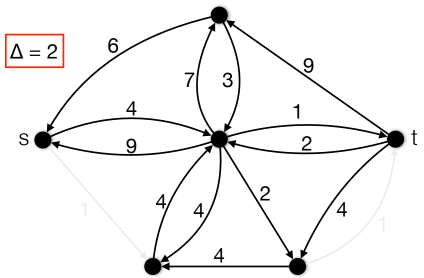

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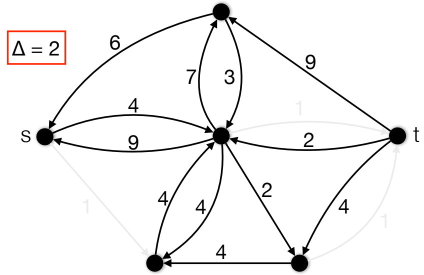

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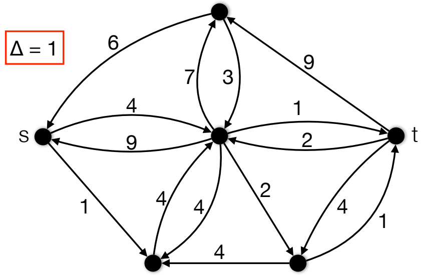

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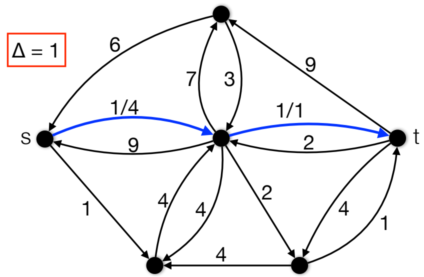

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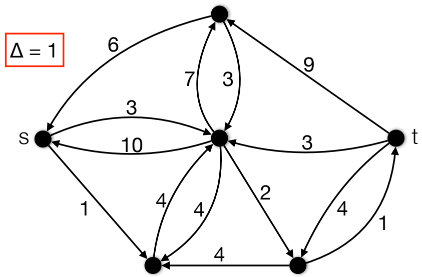

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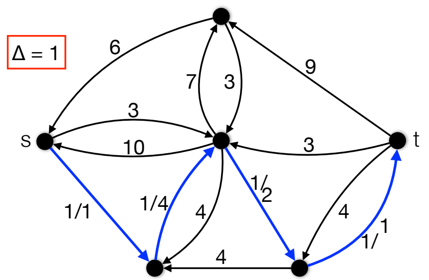

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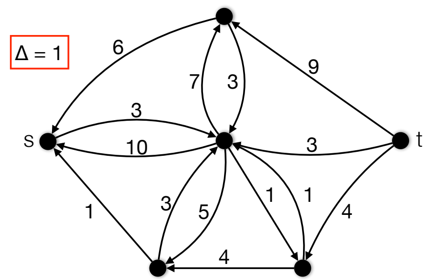
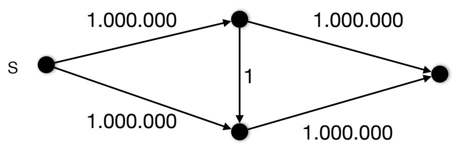


- Stop when no more augmenting paths in $G_f(1)$.

Scaling algorithm



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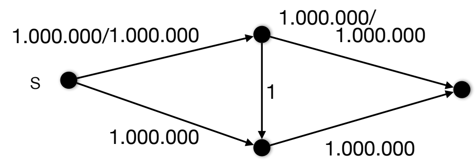

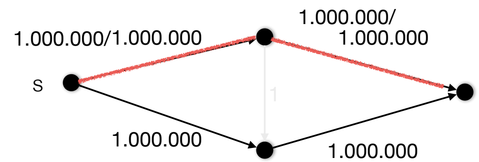

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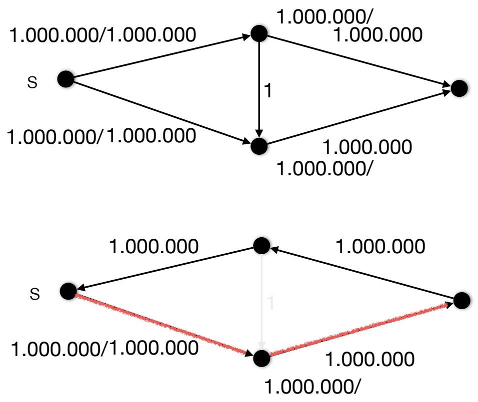

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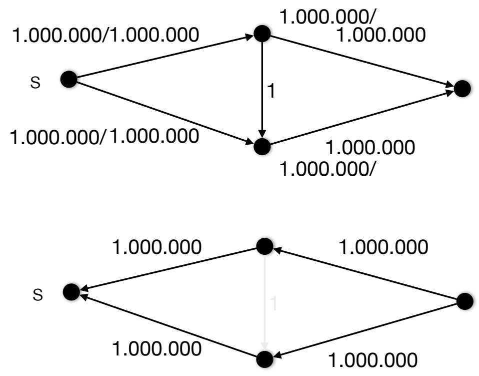

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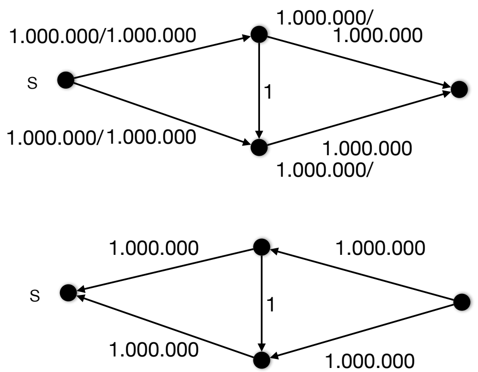
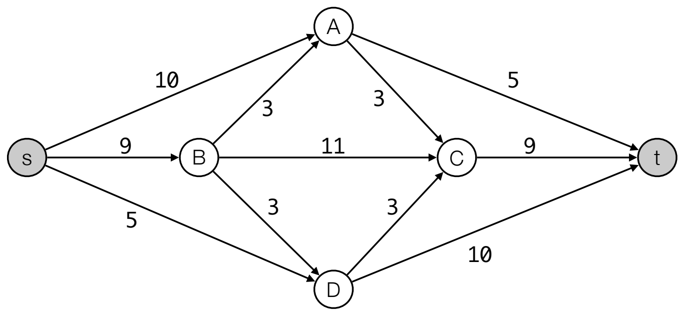


Exercise



Scaling algorithm

- Running time: O(m² log C), where C is the largest capacity out of s.

- Lemma 1. *Number of scaling phases: 1 + ⌈lg C⌉*

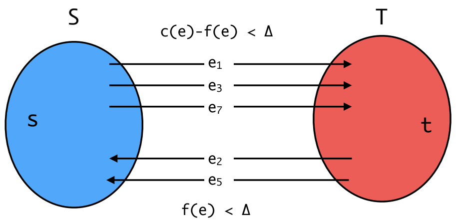
- Prove: number of augmentation paths found in a scaling phase is at most 2m.

- Lemma 2. *Let f the flow when Δ-scaling phase ends, and let f* be the maximum flow. Then v(f*) ≤ v(f) + mΔ.*

Scaling algorithm

- Lemma 2. *Let f the flow when Δ-scaling phase ends, and let f*be the maximum flow. Then v(f*) ≤ v(f) + mΔ.*

- By the end of the phase there is a cut c(S,T) ≤ v(f) + mΔ.



$$c(S,T) = c(e_1) + c(e_3) + c(e_7)$$

$$v(f) = f(e_1) + f(e_3) + f(e_7) - f(e_2) - f(e_5)$$

$$\begin{aligned} c(S,T) - v(f) &= c(e_1) + c(e_3) + c(e_7) - f(e_1) - f(e_3) - f(e_7) + f(e_2) + f(e_5) \\ &= c(e_1) - f(e_1) + c(e_3) - f(e_3) + c(e_7) - f(e_7) + f(e_2) + f(e_5) \\ &< \Delta + \Delta + \Delta + \Delta + \Delta = 5\Delta \end{aligned}$$

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- Lemma 2. *Let f the flow when Δ-scaling phase ends, and let f* be the maximum flow. Then v(f*) ≤ v(f) + mΔ.*

- Lemma 3. *The number of augmentations in a scaling phase is at most 2m.*
  - First phase: can use each edge out of s in at most one augmenting path.
  - f flow at the end of previous phase.
  - Used Δ' = 2Δ in last round.
  - Lemma 2: v(f*) ≤ v(f) + mΔ' = v(f) + 2mΔ.
  - "Leftover flow" to be found ≤ 2mΔ.
  - Each agumentation in a Δ-scaling phase augments flow with at least Δ.

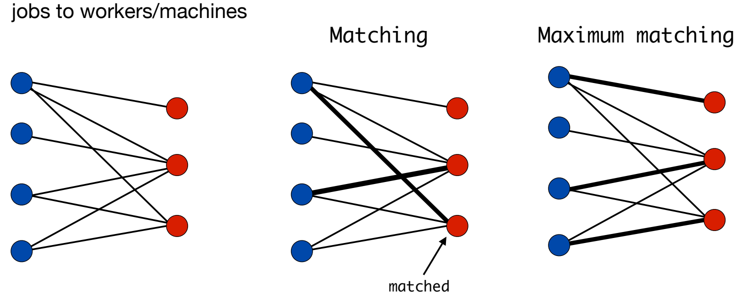
Maximum flow algorithms

- Edmonds-Karp: O(m²n)
- Scaling: O(m² log C)
- Ford-Fulkerson: O(m v(f)).
- Preflow-push O(n³)
- Other algorithms: O(mn log n) or O(min(n^{2/3}, m^{1/2})m log n log U).

Maximum Bipartite Matching

- Bipartite graph: Can color vertices red and blue such that all edges have a red and a blue endpoint.
- Matching: Subset of edges M ⊆ E such that no edges in M share an endpoint.
- Maximum matching: matching of maximum cardinality.

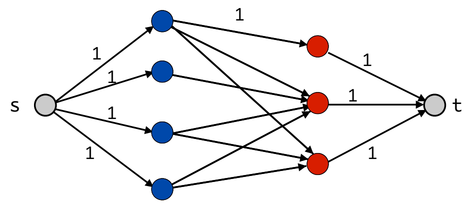
- Applications:
  - planes to routes
  - jobs to workers/machines



Maximum Bipartite Matching

- Bipartite graph: Can color vertices red and blue such that all edges have a red and a blue endpoint.
- Matching: Subset of edges M ⊆ E such that no edges in M share an endpoint.
- Maximum matching: matching of maximum cardinality.

- Solve via flow:



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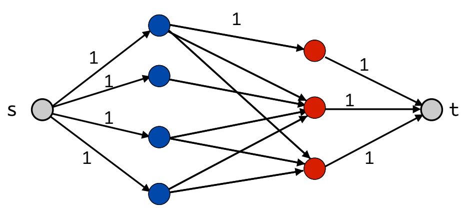

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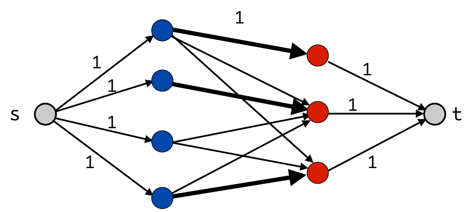

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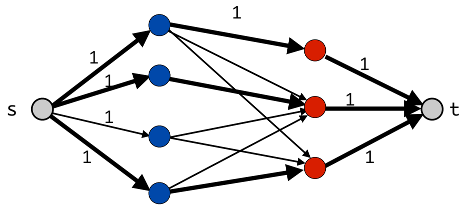

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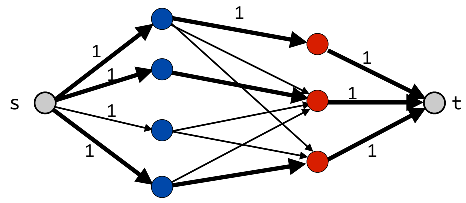
  - Flow of value v(f) => matching of size v(f)



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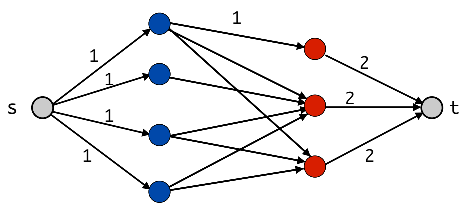
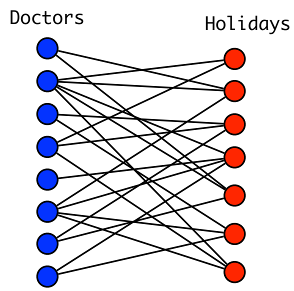
- Solve via flow:
- Can generalize to general matchings



Scheduling of doctors

- X doctors, Y holidays, each doctor should work at at most 1 holiday, each doctor is available at some of the holidays.



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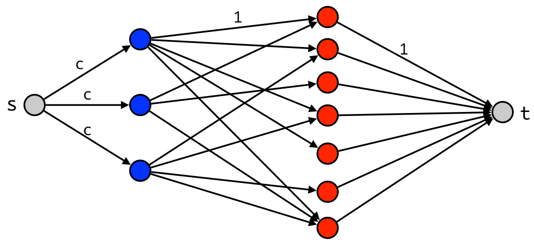

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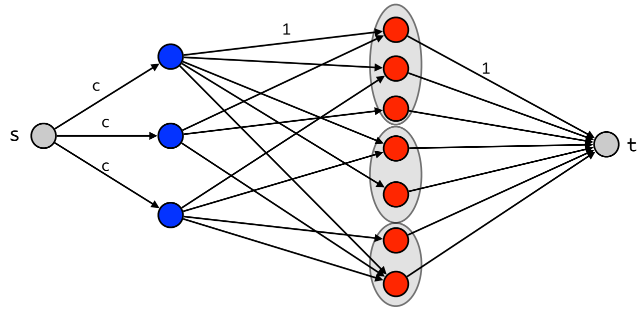

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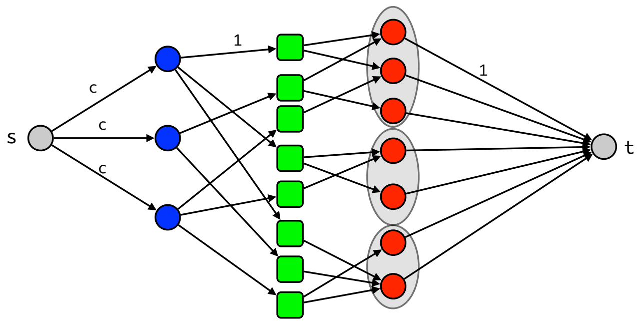
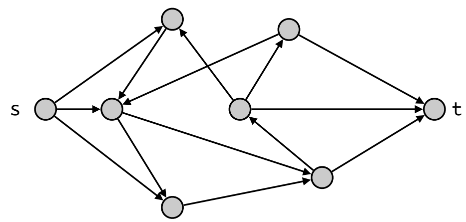


Edge Disjoint paths

- Problem: Find maximum number of edge-disjoint paths from s to t.
- Two paths are edge-disjoint if they have no edge in common.



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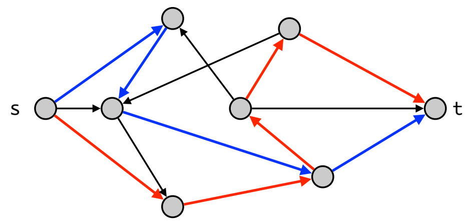
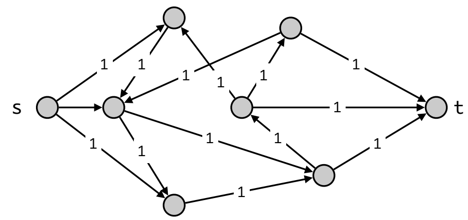


Edge Disjoint Paths

- Reduction to max flow: assign capacity 1 to each edge.

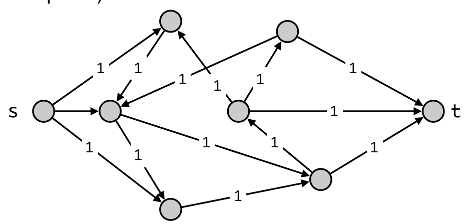


- Thm. Max number of edge-disjoint s-t paths is equal to the value of a maximum flow.
  - Suppose there are k edge-disjoint paths: then there is a flow of k (let all edges on the paths have flow 1).
  - Other way (graph theory course).

- Ford-Fulkerson: v(f) ≤ n (no multiple edges and therefore at most n edges out of s) => running time O(nm).

Network Connectivity

- Network connectivity. Find minimum number of edges whose removal disconnects t from s (destroys all s-t paths).



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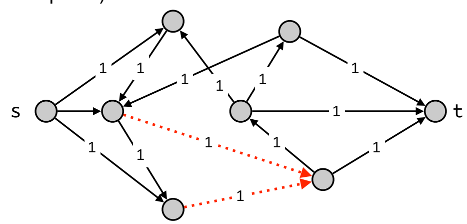


- Set all capacities to 1 and find minimum cut.
- Thm. (Menger) The maximum number of edge-disjoint s-t paths is equal to the minimum number of edges whose removal disconnects t from s.

Baseball elimination

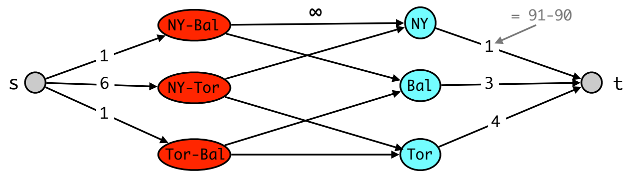
Team	Wins	Games left	Against			
			NY	Bal	Tor	Bos
New York	92	2	-	1	1	0
Baltimore	91	3	1	-	1	1
Toronto	91	3	1	1	-	1
Boston	90	2	0	1	1	-

- Question: Can Boston finish in first place (or in tie of first place)?
- No: Boston must win both its remaining 2 and NY must loose. But then Baltimore and Toronto both beat NY so winner of Baltimore-Toronto will get 93 points.
- Other argument: Boston can finish with at most 92. Cumulatively the other three teams have 274 wins currently and their 3 games against each other will give another 3 points => 277. 277/3 = 92,33333 => one of them must win > 92.

Baseball elimination

Team	Wins	Games left	Against			
			NY	Bal	Tor	Bos
New York	90	11	-	1	6	4
Baltimore	88	6	1	-	1	4
Toronto	87	11	6	1	-	4
Boston	79	12	4	4	4	-

- Question: Can Boston finish in first place (or in tie of first place)?



Boston can get at most 79 + 12 = 91 points

- Boston cannot win (or tie) ⇔ max s-t flow < 8.

Node capacities

- Capacities on nodes.