Dynamic Programming

Algorithm Design 6.1, 6.2, 6.4

Thank you to Kevin Wayne for inspiration to slides

Applications

- In class (today and next time)

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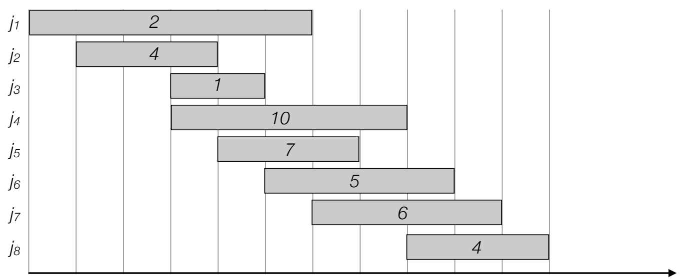
  - Weighted interval scheduling
    - Set of weighted intervals with start and finishing times
    - Goal: find maximum weight subset of non-overlapping intervals



Applications

- Today and next time
  - Weighted interval scheduling
  - Subset Sum and Knapsack
    - Set of items each having a weight and a value
    - Knapsack with a bounded capacity
    - Goal: fill knapsack so as to maximise the total value.

value	10	8	2	5	15	4
weight	2	3	1	2	5	4

Capacity 8

Applications

- Today and next time
  - Weighted interval scheduling
  - Subset Sum and Knapsack
  - Sequence alignment
    - Given two strings A and B how many edits (insertions, deletions, relabelings) is needed to turn A into B?

```
A C A A G T C          A C A A - G T C
- C A T G T -          - C A - T G T -

1 mismatch, 2 gaps      0 mismatches, 4 gaps
```


Dynamic Programming

- Greedy. Build solution incrementally, optimizing some local criterion.
- Divide-and-conquer. Break up problem into independent subproblems, solve each subproblem, and combine to get solution to original problem.
- Dynamic programming. Break up problem into overlapping subproblems, and build up solutions to larger and larger subproblems.
  - Can be used when the problem have "optimal substructure":
    - *Solution can be constructed from optimal solutions to subproblems*
    - *Use dynamic programming when subproblems overlap.*

Computing Fibonacci numbers

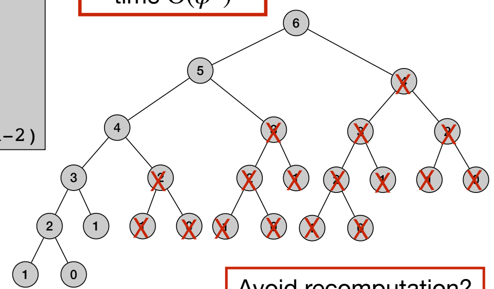
- Fibonacci numbers:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

- First try:

```
Fib(n)
if n = 0
  return 0
else if n = 1
  return 1
else
  return Fib(n-1) + Fib(n-2)
```

time $\Theta(\phi^n)$



Avoid recomputation?

Memoized Fibonacci numbers

- Fibonacci numbers:

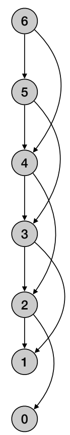
$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

- Remember already computed values:

```
for j=1 to n
  F[j] = null
Mem-Fib(n)

Mem-Fib(n)
if n = 0
  return 0
else if n = 1
  return 1
else
  if F[n] is empty
    F[n] = Mem-Fib(n-1) + Mem-Fib(n-2)
  return F[n]
```

time $\Theta(n)$



Bottom-up Fibonacci numbers

- Fibonacci numbers:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

- Remember already computed values:

```
Iter-Fib(n)
F[0] = 0
F[1] = 1
for i = 2 to n
  F[i] = F[i-1] + F[i-2]
return F[n]
```

time $\Theta(n)$

space $\Theta(n)$

Bottom-up Fibonacci numbers - save space

- Fibonacci numbers:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

- Remember last two computed values:

```
Iter-Fib(n)
previous = 0
current = 1
for i = 1 to n
  next = previous + current
  previous = current
  current = next
return current
```

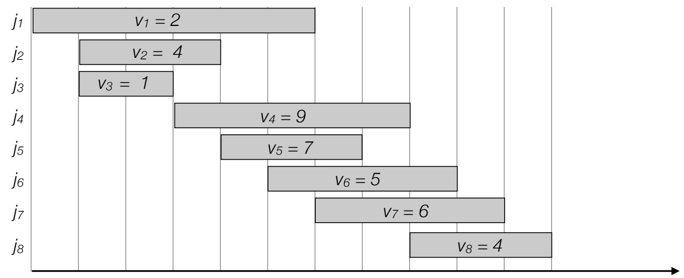
time $\Theta(n)$

space $\Theta(1)$

Weighted Interval Scheduling

Weighted interval scheduling

- Weighted interval scheduling problem
  - n jobs (intervals)
  - Job i starts at s_i, finishes at f_i and has weight/value v_i.
  - Goal: Find maximum weight subset of non-overlapping (compatible) jobs.



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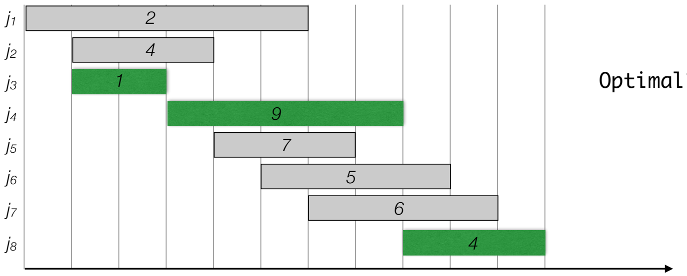
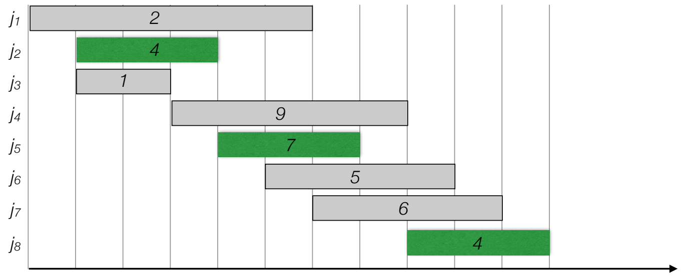


Optimal?

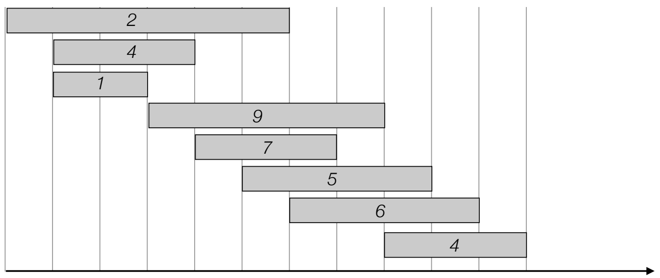
Weighted interval scheduling

- Weighted interval scheduling problem
  - n jobs (intervals)
  - Job i starts at s_i, finishes at f_i and has weight/value v_i.
  - Goal: Find maximum weight subset of non-overlapping (compatible) jobs.



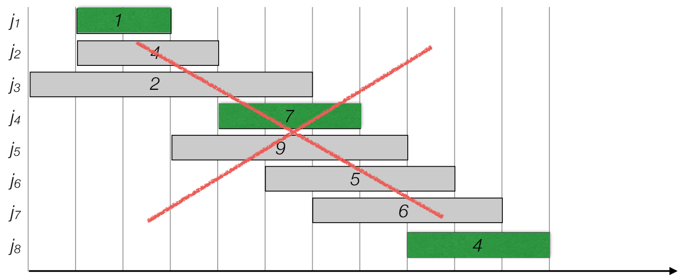
Weighted interval scheduling

- Label/sort jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$



Weighted interval scheduling

- Label/sort jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$
- ~~Greedy?~~

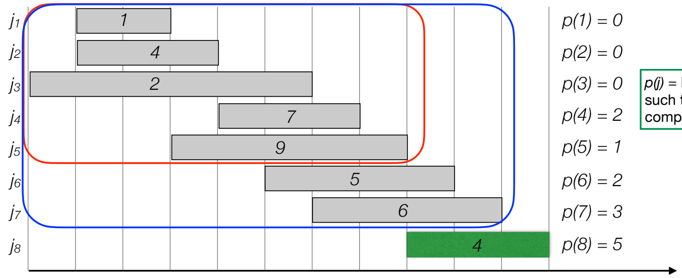


Weighted interval scheduling

- Label/sort jobs by finishing time: $f_1 \le f_2 \le \ldots \le f_n$
- Optimal solution OPT:
  - Case 1. OPT selects last job

    OPT = v_n + optimal solution to subproblem on the subset of jobs ending before job n starts
  - Case 2. OPT does not select last job

    OPT = optimal solution to subproblem on 1,…,n-1



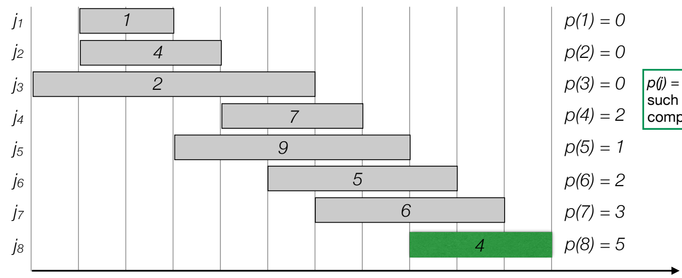
p(j) = largest index i < j such that job i is compatible with j.

Weighted interval scheduling

- Label/sort jobs by finishing time: $f_1 \le f_2 \le \ldots \le f_n$
- Optimal solution OPT:
  - Case 1. OPT selects last job

    OPT = v_n + optimal solution to subproblem on 1,…,p(n)
  - Case 2. OPT does not select last job

    OPT = optimal solution to subproblem on 1,…,n-1



p(j) = largest index i < j such that job i is compatible with j.

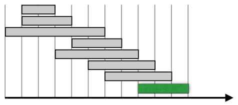
Weighted interval scheduling

- OPT(j) = value of optimal solution to the problem consisting job requests 1,2,...j.
  - Case 1. OPT(j) selects job j

    OPT(j) = v_j + optimal solution to subproblem on 1,…,p(j)
  - Case 2. OPT(j) does not select job j

    OPT = optimal solution to subproblem 1,…j-1
- Recurrence:

$$OPT(j) = \begin{cases} 0 & \texttt{if } j = 0 \\ \max\{v_j + OPT(p(j)), OPT(j-1)\} & \texttt{otherwise} \end{cases}$$

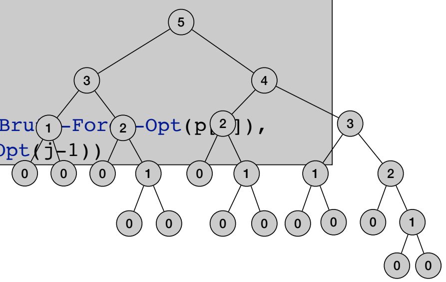


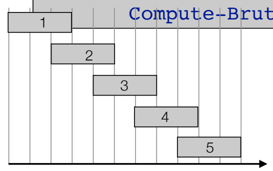
Weighted interval scheduling: brute force

$$OPT(j) = \begin{cases} 0 & \texttt{if } j = 0 \\ \max\{v_j + OPT(p(j)), OPT(j-1)\} & \texttt{otherwise} \end{cases}$$

```
Input: n, s[1..n], f[1..n], v[1..n]

Sort jobs by finish time so that f[1] ≤ f[2]≤ … ≤ f[n]
Compute p[1], p[2], …, p[n]
Compute-BruteForce-Opt(n)

Compute-Brute-Force-Opt(j)
if j = 0
  return 0
else
  return max(v[j] + Compute-Brute-Force-Opt(p[j]),
         Compute-Brute-Force-Opt(j-1))
```

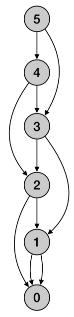
time $\Theta(2^n)$





Weighted interval scheduling: memoization

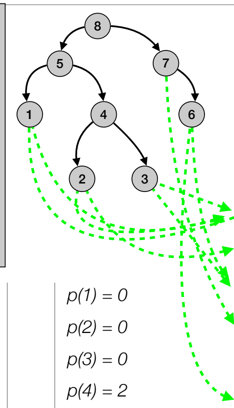
```
Input: n, s[1..n], f[1..n], v[1..n]

Sort jobs by finish time so that f[1] ≤ f[2]≤ … ≤ f[n]
Compute p[1], p[2], …, p[n]

for j=1 to n
  M[j] = null
M[0] = 0.
Compute-Memoized-Opt(n)

Compute-Memoized-Opt(j)
if M[j] is empty
  M[j] = max(v[j] + Compute-Memoized-Opt(p[j]),
         Compute-Memoized-Opt(j-1))
return M[j]
```



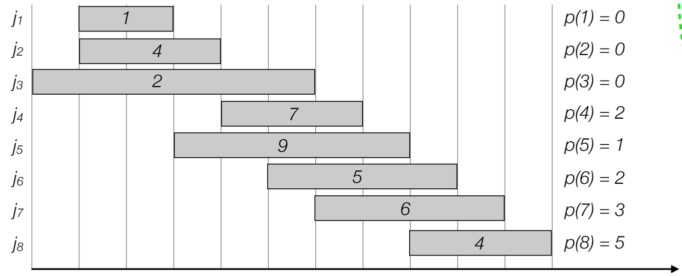
- Running time O(n log n):
  - Sorting takes O(n log n) time.
  - Computing p(n): O(n log n) - use log n time to find each p(i).
  - Each subproblem solved once.
  - Time to solve a subproblem constant.
- Space O(n)

Weighted interval scheduling: memoization

```
Input: n, s[1..n], f[1..n], v[1..n]

Sort jobs by finish time so that f[1] ≤ f[2]≤ … ≤ f[n]
Compute p[1], p[2], …, p[n]

for j=1 to n
  M[j] = empty
M[0] = 0.
Compute-Memoized-Opt(n)

Compute-Memoized-Opt(j)
if M[j] is empty
  M[j] = max(v[j] + Compute-Memoized-Opt(p[j]),
         Compute-Memoized-Opt(j-1))
return M[j]
```

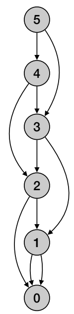


i	M[i]
0	0
1	1
2	4
3	4
4	11
5	11
6	11
7	11
8	15



Weighted interval scheduling: bottom-up

```
Compute-Bottom-Up-Opt(n, s[1..n], f[1..n], v[1..n])

Sort jobs by finish time so that f[1] ≤ f[2]≤ … ≤ f[n]
Compute p[1], p[2], …, p[n]

M[0] = 0.
for j=1 to n
  M[j] = max(v[j] + M(p[j]), M(j-1))
return M[n]
```



- Running time O(n log n):
  - Sorting takes O(n log n) time.
  - Computing p(n): O(n log n)
  - For loop: O(n) time
    - Each iteration takes constant time.
- Space O(n)

Weighted interval scheduling: find solution

```
Find-Solution(j)
if j=0
  Return emptyset
else if M[j] > M[j-1]
  return {j} ∪ Find-Solution(p[j])
else
  return Find-Solution(j-1)
```

i	M[i]
0	0
1	1
2	4
3	4
4	11
5	11
6	11
7	11
8	15

Solution = 8 , 4 , 2