## References and Reading

- KT Chapter 12, Sections 12(.0), 12.1, 12.2, and 12.12.

Section 12.12 contains an excellent introduction to basic probability theory needed for the course. (It is Chapter 13 in the American edition).

## Exercises

**1** [$w$] **Probability Basics** Let $E$ and $F$ be two events such that $\Pr(E \mid F) = \Pr(E)$ and $\Pr(F \mid E) = \Pr(F)$. Show that $\Pr(E \cap F) = \Pr(E) \cdot \Pr(F)$.

**2** [$w$] **Contention Resolution** Run the contention resolution protocol with 4 processes. Use two coins or a random number generator (webpages with random number generators are readily available) to simulate the random choices. How many rounds do you need before all processes have successfully accessed the database? How does this compare with the theoretical results?

**3** **Minimum cut** Consider the tiny graph example for contraction algorithm covered in the lecture. Solve the following exercises.

**3.1** Show that there is a choice of contractions that lead to a cut that is not minimum.

**3.2** Compute the probability that the contraction algorithm returns the minimum cut.

**4** **Fast Contraction Algorithm** Show how to implement the contraction algorithm for minimum cut efficiently. *Note:* You can compute a random permutation of $N$ numbers in $O(N)$ time.

**5** **Contraction Algorithm Analysis** Consider the probability analysis of success for the contraction algorithm. The following derivation leads to the desired result.

$$\begin{aligned}
\Pr(E_{n-2} \cap \cdots \cap E_1) &= \Pr(E_{n-3} \cap \cdots \cap E_1) \cdot \Pr(E_{n-2} \mid E_{n-3} \cap \cdots \cap E_1) && (1)\\
&= \Pr(E_1) \cdot \Pr(E_2 \mid E_1) \cdots \Pr\left(E_{j+1} \mid E_j \cap \cdots \cap E_1\right) \cdots \Pr(E_{n-2} \mid E_{n-3} \cap \cdots \cap E_1) && (2)\\
&\geq \left(1 - \frac{2}{n}\right) \cdot \left(1 - \frac{2}{n-1}\right) \cdot \left(1 - \frac{2}{n-2}\right) \cdots \left(1 - \frac{2}{3}\right) && (3)\\
&= \left(\frac{n-2}{n}\right) \cdot \left(\frac{n-3}{n-1}\right) \cdot \left(\frac{n-4}{n-2}\right) \cdots \left(\frac{2}{4}\right) \cdot \left(\frac{1}{3}\right) && (4)\\
&= \left(\frac{2}{n(n-1)}\right) \geq \frac{2}{n^2} && (5)
\end{aligned}$$

Solve the following exercises.

**5.1** Show (1) in two ways: (a) using the formula for conditional probability and (b) by interpreting what the lefthand side and righthand side represent.

**5.2** Show (2). *Hint:* use the result from exercise **5.1**.

**5.3** Plug in the bounds for each factor to show (3).

**5.4** Show (4).

**5.5** Show (5).

**6 Majority** Given is a sequence $x_1, x_2, \dots, x_n$ of $n$ integers. The sequence has a *majority element* $t$, if a number $t$ occurs strictly more than $n/2$ times in the sequence. For example the sequence $1, 2, 3, 1, 2, 2, 2$ has a majority element, namely 2, whereas the sequence $2, 2, 1, 2, 3, 3$ has no majority element.

Here is a randomized algorithm for this problem: Randomly pick an number $x_i$ from the sequence. Then check whether it occurs more than $n/2$ times in the sequence. If so it is a majority element. Otherwise the algorithm answers that there is no majority element. Solve the following exercises.

**6.1** What is running time of the algorithm?

**6.2** Can the algorithm return a majority number, if the sequence does not have one? Justify your answer.

**6.3** Can the algorithm claim that there is no majority number, even though the sequence does have one? Justify your answer.

**6.4** Determine an upper bound on the probability for incorrect answers.