# Randomized algorithms II

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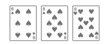
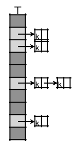
Thank you to Kevin Wayne and Philip Bille for inspiration to slides

## Randomized algorithms

- Last week
  - Contention resolution
  - Global minimum cut
- Today
  - Expectation of random variables
    - Guessing cards
  - Hash functions and hash tables





# Random Variables and Expectation

## Random variables

- A random variable is an entity that can assume different values.
- The values are selected “randomly”; i.e., the process is governed by a probability distribution.
- Examples: Let X be the random variable “number shown by dice”.
  - X can take the values 1, 2, 3, 4, 5, 6.
  - If it is a fair dice then the probability that X = 1 is 1/6:
    - Pr[X=1] = 1/6.
    - Pr[X=2] = 1/6.
    - …

## Expected values

- Let X be a random variable with values in $\{x_1,\ldots x_n\}$, where $x_i$ are numbers.
- The expected value (expectation) of X is defined as

$$E[X] = \sum_{j=1}^{n} x_j \cdot \Pr[X = x_j]$$

- The expectation is the theoretical average.
- Example:
  - X = random variable "number shown by dice"

$$E[X] = \sum_{j=1}^{6} j \cdot \Pr[X = j] = (1 + 2 + 3 + 4 + 5 + 6) \cdot \frac{1}{6} = 3.5$$

## Waiting for a first succes

- Coin flips. Coin is heads with probability $p$ and tails with probability $1 - p$. How many independent flips X until first heads?
  - Probability of $X = j$? (first succes is in round $j$)

$$\Pr[X = j] = (1-p)^{j-1} \cdot p$$

  - Expected value of $X$:

$$E[X] = \sum_{j=1}^{\infty} j \cdot \Pr[X = j] = \sum_{j=1}^{\infty} j \cdot (1-p)^{j-1} \cdot p = \frac{p}{1-p} \sum_{j=1}^{\infty} j \cdot (1-p)^j$$

$$= \frac{p}{1-p} \cdot \frac{1-p}{p^2} = \frac{1}{p}$$

$$\sum_{k=0}^{\infty} k \cdot x^k = \frac{x}{(1-x)^2} \quad \text{for} \quad x < 1.$$

## Properties of expectation

- If we repeatedly perform independent trials of an experiment, each of which succeeds with probability $p > 0$, then the expected number of trials we need to perform until the first succes is $1/p$.
- If $X$ is a 0/1 random variable, then $E[X] = \Pr[X = 1]$.
- Linearity of expectation: For two random variables X and Y we have

$$E[X + Y] = E[X] + E[Y]$$

## Guessing cards

- Game. Shuffle a deck of $n$ cards; turn them over one at a time; try to guess each card.
- Memoryless guessing. Can't remember what's been turned over already. Guess a card from full deck uniformly at random.

- Claim. *The expected number of correct guesses is 1.*
  - $X_i = 1$ if $i^{th}$ guess correct and zero otherwise.
  - $X =$ the correct number of guesses $= X_1 + \ldots + X_n$.
  - $E[X_i] = \Pr[X_i = 1] = 1/n$.
  - $E[X] = E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n] = 1/n + \cdots + 1/n = 1.$

## Guessing cards

- Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.
- Guessing with memory. Guess a card uniformly at random from cards not yet seen.
- Claim. *The expected number of correct guesses is* $\Theta(\log n)$.
  - $X_i = 1$ if $i^{th}$ guess correct and zero otherwise.
  - $X =$ the correct number of guesses $= X_1 + \ldots + X_n$.
  - $E[X_i] = \Pr[X_i = 1] = 1/(n-i+1)$.
  - $E[X] = E[X_1] + \cdots + E[X_n] = 1/n + \cdots + 1/2 + 1/1 = H_n$.

$\ln n < H(n) < \ln n + 1$

## Coupon collector

- Coupon collector. Each box of cereal contains a coupon. There are $n$ different types of coupons. Assuming all boxes are equally likely to contain each coupon, how many boxes before you have at least 1 coupon of each type?
- Claim. *The expected number of steps is* $\Theta(n \log n)$.
  - Phase $j$ = time between $j$ and $j+1$ distinct coupons.
  - $X_j$ = number of steps you spend in phase $j$.
  - $X$ = number of steps in total = $X_0 + X_1 + \cdots + X_{n-1}$.
  - $E[X_j] = n/(n-j)$.
  - The expected number of steps:

$$E[X] = E[\sum_{j=0}^{n-1} X_j] = \sum_{j=0}^{n-1} E[X_j] = \sum_{j=0}^{n-1} n/(n-j) = n \cdot \sum_{i=1}^{n} 1/i = n \cdot H_n.$$

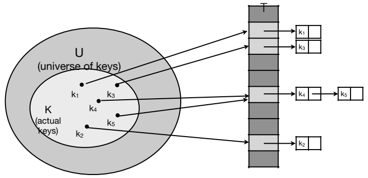
## Hashing

## Dictionaries

- Dictionary problem. Maintain a dynamic set of S ⊆ U subject to the following operations:
  - Lookup(x): return true if x ∈ S and false otherwise
  - Insert(x): Set S = S ∪ {x}
  - Delete(x): Set S = S \ {x}
- Universe size. Typically |U| = 2^64 and |S| << |U|.
- Satellite information. Information associated with each element.
- Goal. A compact data structure with fast operations.
- Applications. Many! A key component in other data structures and algorithms.

## Chained Hashing

- Chained hashing [Dumey 1956].
  - n = |S|.
  - Hash function. Pick some crazy, chaotic, random function h that maps U to {0, …, m-1}, where m = Θ(n).
  - Initialise an array A[0, …, m-1].
  - A[i] stores a linked list containing the keys in S whose hash value is i.




## Uniform random hash functions

- *E.g. h(x) = x mod 11*. Not crazy, chaotic, random.
- Suppose |U| ≥ n²: For any hash function h there will be a set S of n elements that all map to the same position!

  => we end up with a single linked list.
- Solution: randomization.
  - For every element u ∈ U: select h(u) uniformly at random in {0, …, m-1} independently from all other choices.
- Claim. *The probability that h(u) = h(v) for two elements u ≠ v is 1/m.*
- Proof.
  - m² possible choices for the pair of values (h(u),h(v)). All equally likely.
  - Exactly m of these gives a collision.

## Chained Hashing with Random Hash Function

- Expected length of the linked list for h(x)?
- Random variable $L_x$ = length of linked list for x. $L_x = \{y \in S \mid h(y) = h(x)\}$
- Indikator random variable:

$$I_y = \begin{cases} 1 & \text{if } h(x) = h(y) \\ 0 & \text{otherwise} \end{cases} \qquad L_x = \sum_{y \in S} I_y \qquad E[I_y] = \Pr[h(y) = h(x)] = \frac{1}{m} \text{ for } x \neq y.$$

- The expected length of the linked list for x:

$$E[L_x] = E\left[\sum_{y \in S} I_y\right] = \sum_{y \in S} E[I_y] = 1 + \sum_{y \in S \setminus \{x\}} \frac{1}{m} = 1 + (n-1) \cdot \frac{1}{m} = \Theta(1).$$

## Chained Hashing with Random Hash Function

- Constant time and O(n) space for the hash table.
- But:
  - Need O(|U|) space for the hash function.
  - Need a lot of random bits to generate the hash function.
  - Need a lot of time to generate the hash function.
- Do we need a truly random hash function?
- When did we use the fact that h was random in our analysis?

## Chained Hashing with Random Hash Function

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## Universal hash functions

- Universal hashing [Carter and Wegman 1979].
  - Let $H$ be a family of functions mapping $U$ to the set $\{0, \ldots, m-1\}$.
  - $H$ is universal if for any $x, y \in U$, where $x \neq y$, and $h$ chosen uniformly at random in $H$,

$$\Pr[h(x) = h(y)] \leq 1/m\,.$$

- Require that any $h \in H$ can be represented compactly and that we can compute the value $h(u)$ efficiently for any $u \in U$.

## Universal Hashing

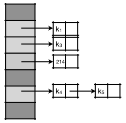
- Positional number systems. For integers x and b, the base-r representation of x is x written in base b.
- Example.
  - $(10)_{10} = (1010)_2$ $(1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0)$
  - $(107)_{10} = (212)_7$ $(2 \cdot 7^2 + 1 \cdot 7^1 + 2 \cdot 7^0)$

## Universal Hashing

- Hash function. Given a prime p and a = $(a_1 a_2 \ldots a_r)_p$, define

$$h_a((x_1 x_2 \ldots x_r)_p) = a_1 x_1 + a_2 x_2 + \ldots + a_r x_r \mod p$$

- Example.
  - p = 7
  - a = $(107)_{10}$ = $(212)_7$
  - x = $(214)_{10}$ = $(424)_7$
  - $h_a(x) = 2 \cdot 4 + 1 \cdot 2 + 2 \cdot 4 \bmod 7 = 18 \bmod 7 = 4$



- Universal family.
  - $H = \{h_a \mid (a_1 a_2 \ldots a_r)_p \in \{0, \ldots, p-1\}^r\}$
  - Choose random hash function from H ~ choose random a.
  - H is universal (analysis next).
  - O(1) time evaluation.
  - O(1) space.
  - Fast construction.

## Uniform Hashing

- Lemma 1. For any prime $p$, any integer $z \neq 0 \mod p$, and any two integers $\alpha, \beta$:

$$\alpha z = \beta z \mod p \quad \Rightarrow \quad \alpha = \beta \mod p.$$

- Proof.
  - Show $(\alpha - \beta)$ is divisible by $p$:
    - $\alpha z = \beta z \mod p \quad \Rightarrow \quad (\alpha - \beta) z = 0 \mod p.$
    - By assumption $z$ not divisible by $p$.
    - Since $p$ is prime $\alpha - \beta$ must be divisible by $p$.
  - Thus $\alpha = \beta \mod p$ as claimed.

## Universal Hashing

- Goal. For random $a = (a_1 a_2 \ldots a_r)_p$, show that if $x \neq y$ then $\Pr[h_a(x) = h_a(y)] \leq 1/p$.
  - Recall: $x = (x_1 x_2 \ldots x_r)_p$ and $y = (y_1 y_2 \ldots y_r)_p$:

$$x \neq y \Leftrightarrow (x_1 x_2 \ldots x_r)_p \neq (y_1 y_2 \ldots y_r)_p \Rightarrow x_j \neq y_j \text{ for some } j.$$

- Lemma 2. Let $j$ be such that $x_j \neq y_j$. Assume the coordinates $a_i$ have been chosen for all $i \neq j$. The probability of choosing $a_j$ such that $h_a(x) = h_a(y)$ is $1/p$.
  - $h_a(x) = h_a(y) \quad \Leftrightarrow \quad \sum_{i=1}^{r} a_i x_i \mod p = \sum_{i=1}^{r} a_i x y_i \mod p \quad \Leftrightarrow \quad a_i(x_j - y_j) = \sum_{i \neq j} a_i(x_i - y_i) \mod p$ = c

    fixed value z ≠ 0

    fixed value since all $a_i$ fixed for i≠j.
  - *There is exactly one value $0 \leq a_j < p$ that satisfies $a_j z = c \mod p$.*
    - Assume there was two such values $a_j$ and $a_j'$.
      - Then $a_j z = a_j' z \mod p$.
      - Lemma 1 $\Rightarrow a_j = a_j' \mod p$. Since $a_j < p$ and $a_j' < p$ we have $a_j = a_j'$.
  - Probability of choosing $a_j$ such that $h_a(x) = h_a(y)$ is $1/p$.

## Universal Hashing

- Lemma 2. Let $j$ be such that $x_j \neq y_j$. Assume the coordinates $a_i$ have been chosen for all $i \neq j$. The probability of choosing $a_j$ such that $h_a(x) = h_a(y)$ is $1/p$.

- Theorem. For random $a = (a_1 a_2 \ldots a_r)_p$, if $x \neq y$ then

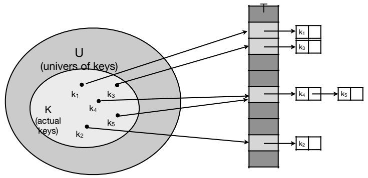
$$\Pr[h_a(x) = h_a(y)] = 1/p\,.$$

- Proof.
  - $E$: the event that $h_a(x) = h_a(y)$.
  - $F_b$: the event that the values $a_i$ for $i \neq j$ gets the sequence of values $b$.
  - Lemma 2 shows that $\Pr[E \;\; F_b] = 1/p$ for all $b$.
  - Thus

$$\Pr[E] = \sum_b \Pr[E \mid F_b] \cdot \Pr[F_b] = \sum_b \frac{1}{p} \cdot \Pr[F_b] = = \frac{1}{p} \sum_b \cdot \Pr[F_b] = \frac{1}{p}$$

## Dictionaries

- Theorem. We can solve the dictionary problem (without special assumptions) in:
  - O(n) space.
  - O(1) expected time per operation (lookup, insert, delete).



## Universal Hashing

- Other universal families.
  - For prime p > 0.

$$h_{a,b}(x) = ax \mod p$$
$$H = \{h_{a,b} \mid a \in \{1,\ldots,p-1\}, b \in \{0,\ldots,p-1\}\}\,.$$

  - Hash function from $k$-bit numbers to $l$-bit numbers.

$$h_a(x) = (ax \mod 2^k) \gg (k-l)$$
$$H = \{h_a \mid a \text{ is an odd integer in } \{1,\ldots,2^k-1\}\}$$