Divide-and-Conquer

Inge Li Gørtz

Thank you to Kevin Wayne for inspiration to slides

Divide-and-Conquer

- Divide -and-Conquer.
  - Break up problem into several parts.
  - Solve each part recursively.
  - Combine solutions to subproblems into overall solution.

- Today
  - Mergesort (recap)
  - Recurrence relations
  - Integer multiplication

Mergesort

Recurrence relations

- T(n) = running time of mergesort on input of size n
- Mergesort recurrence:

$$T(n) \leq \begin{cases} 2T(n/2) + cn & \text{if } n > 2 \\ c & \text{otherwise} \end{cases}$$

- Solving the recurrence:
  - Recursion tree
  - Substitution

Mergesort recurrence: recursion tree

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$$cn$$

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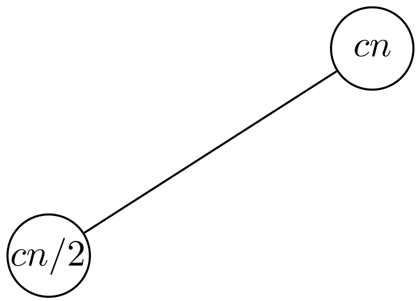

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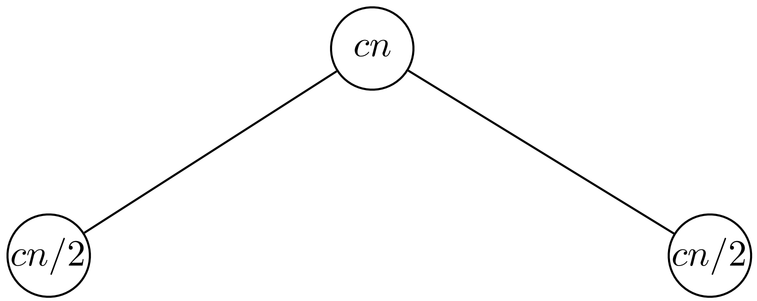

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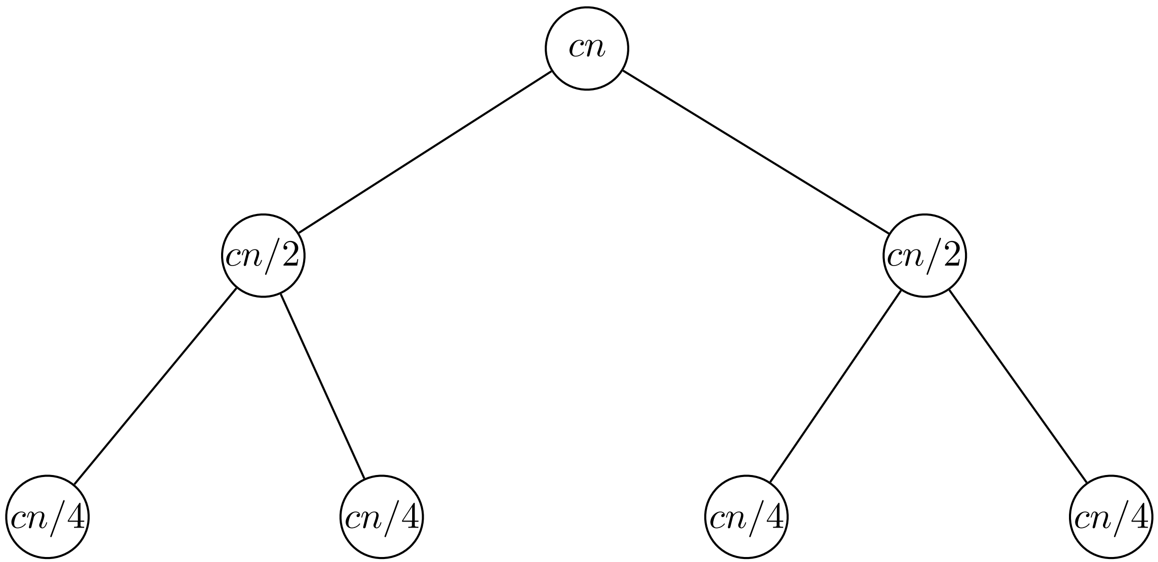

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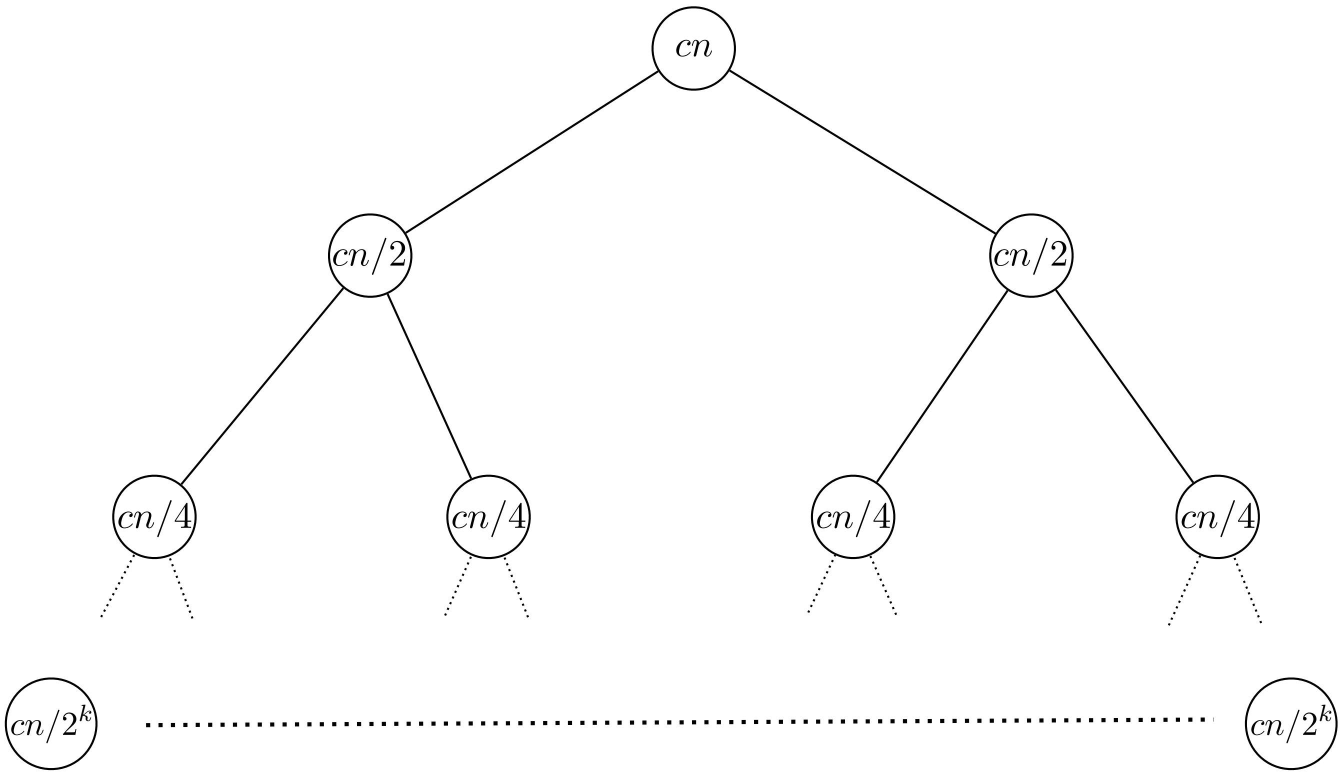

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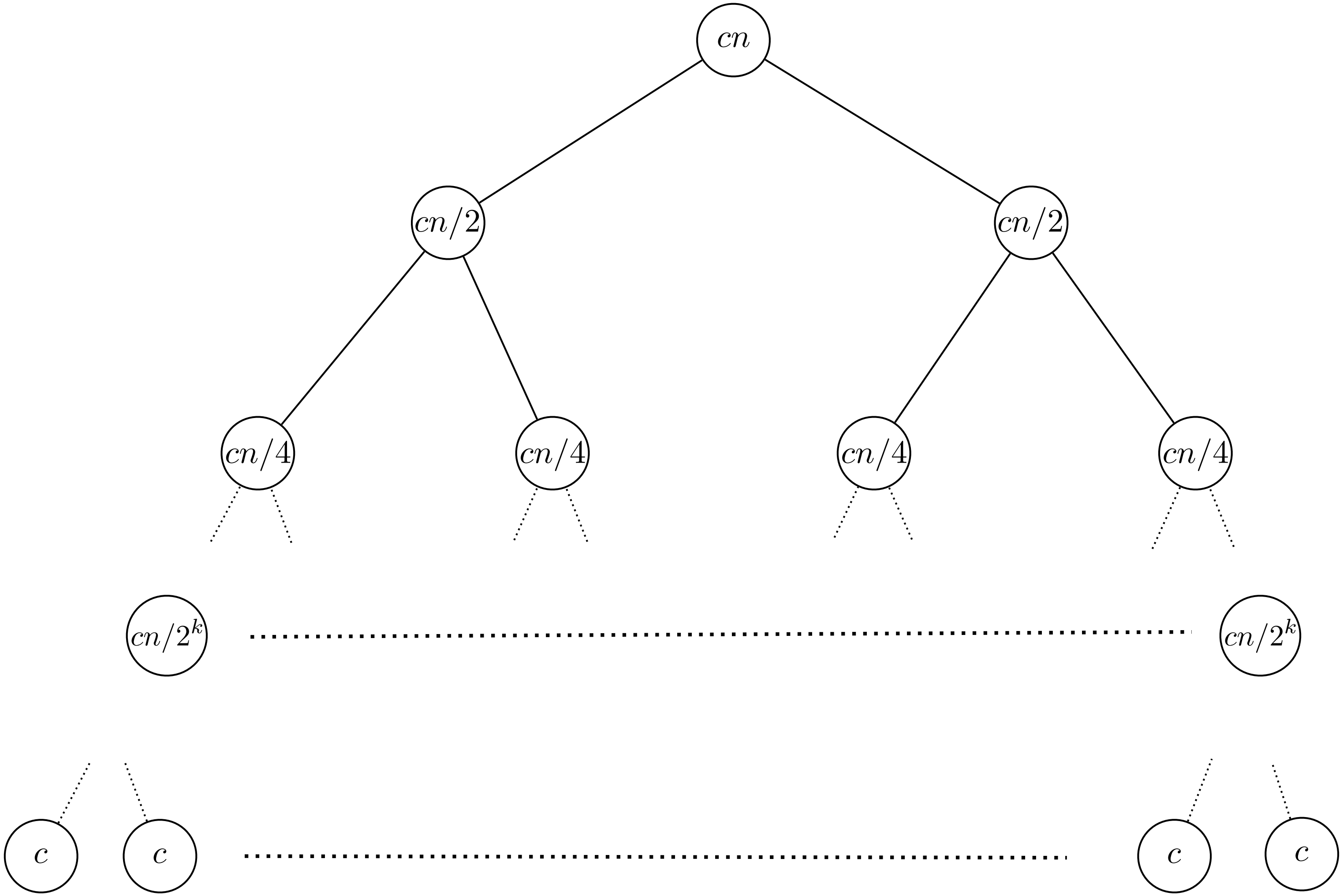

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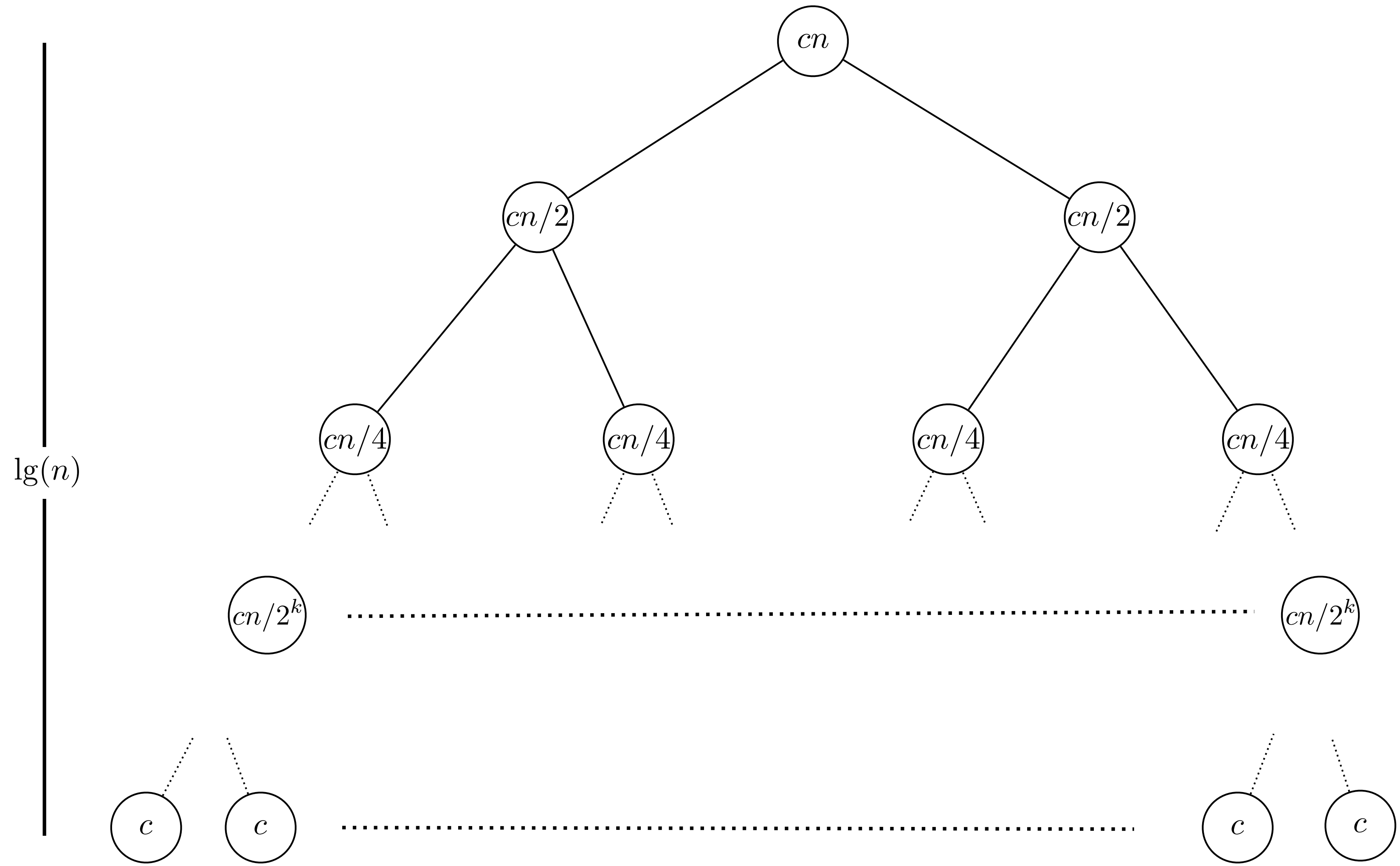

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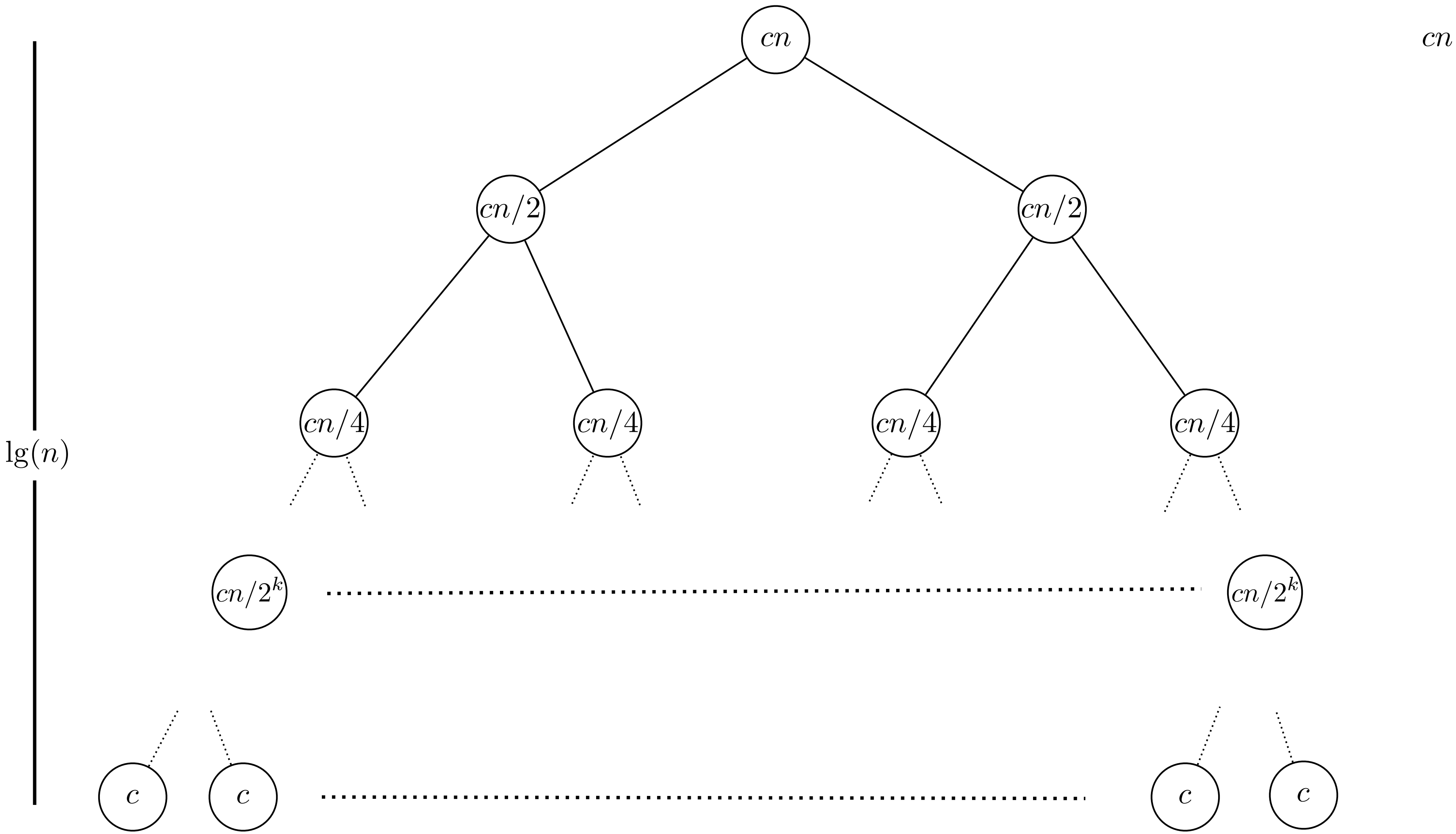

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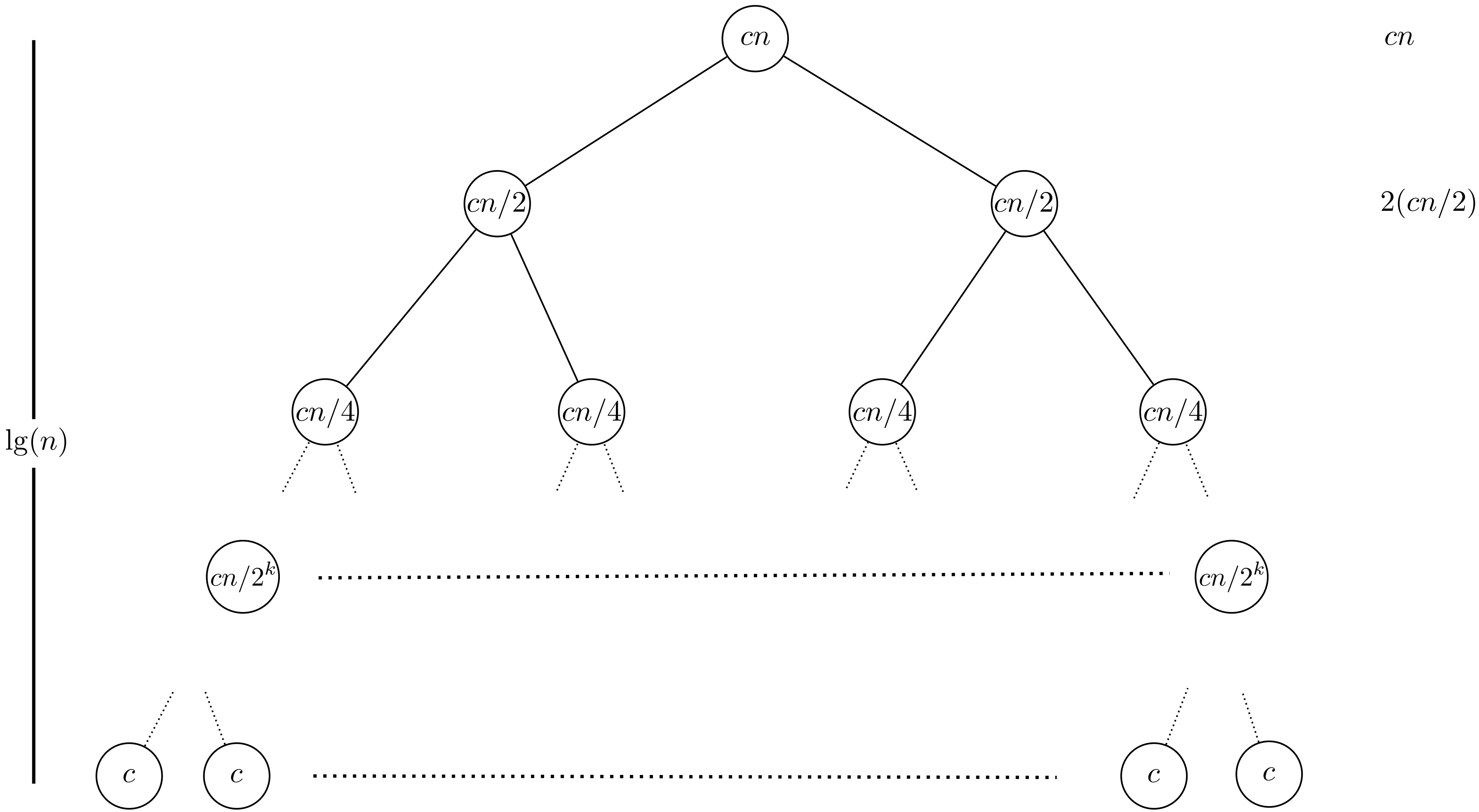

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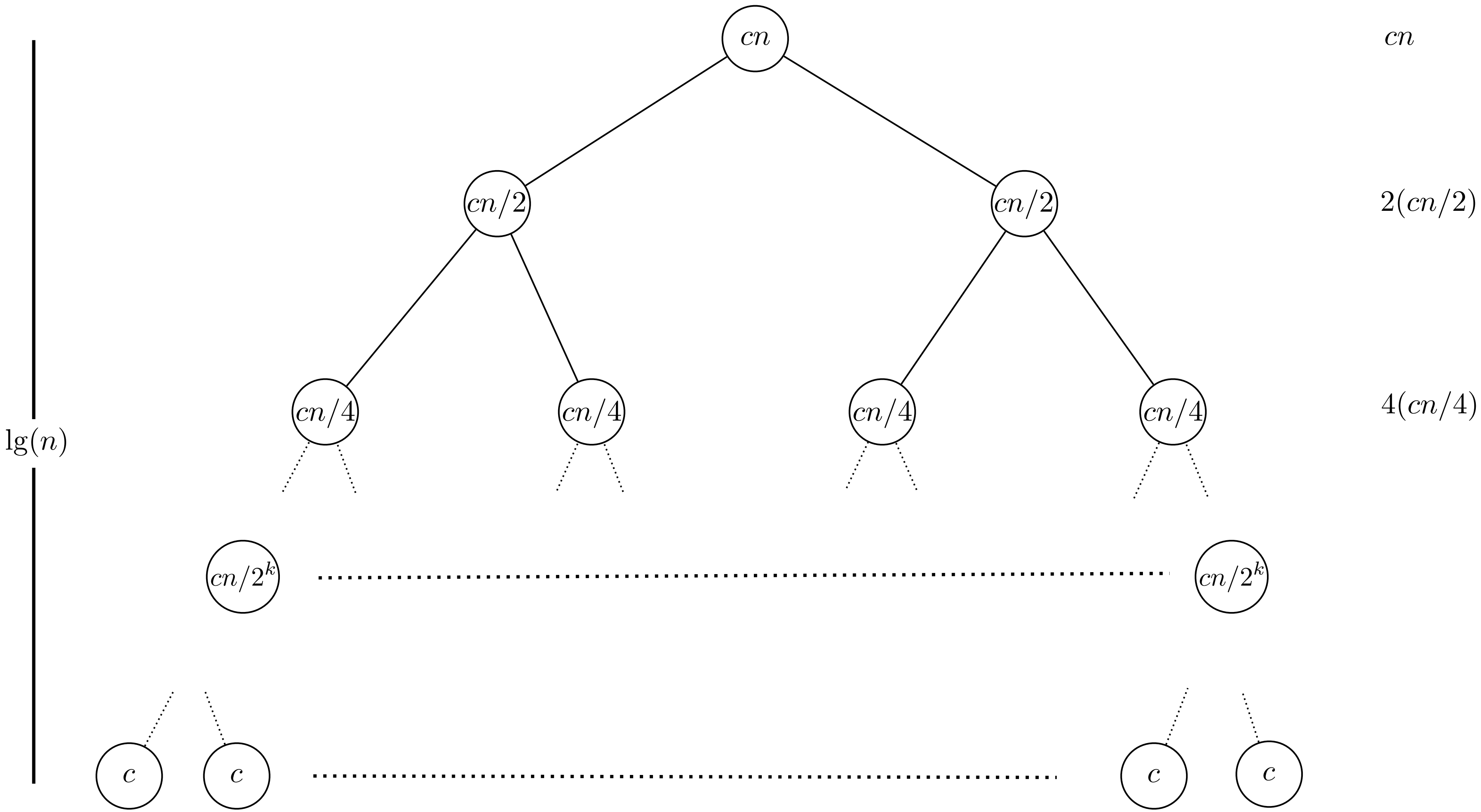

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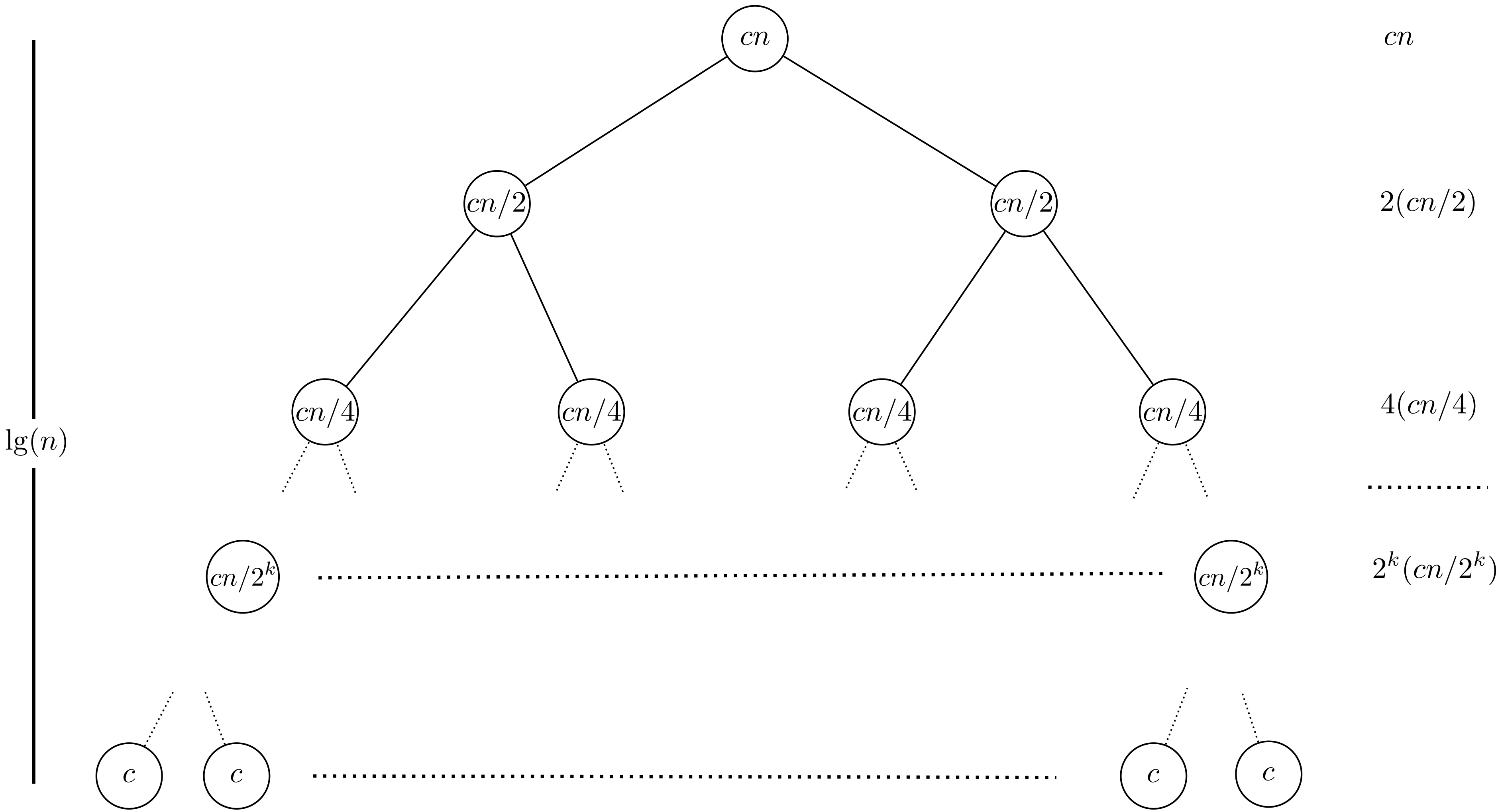

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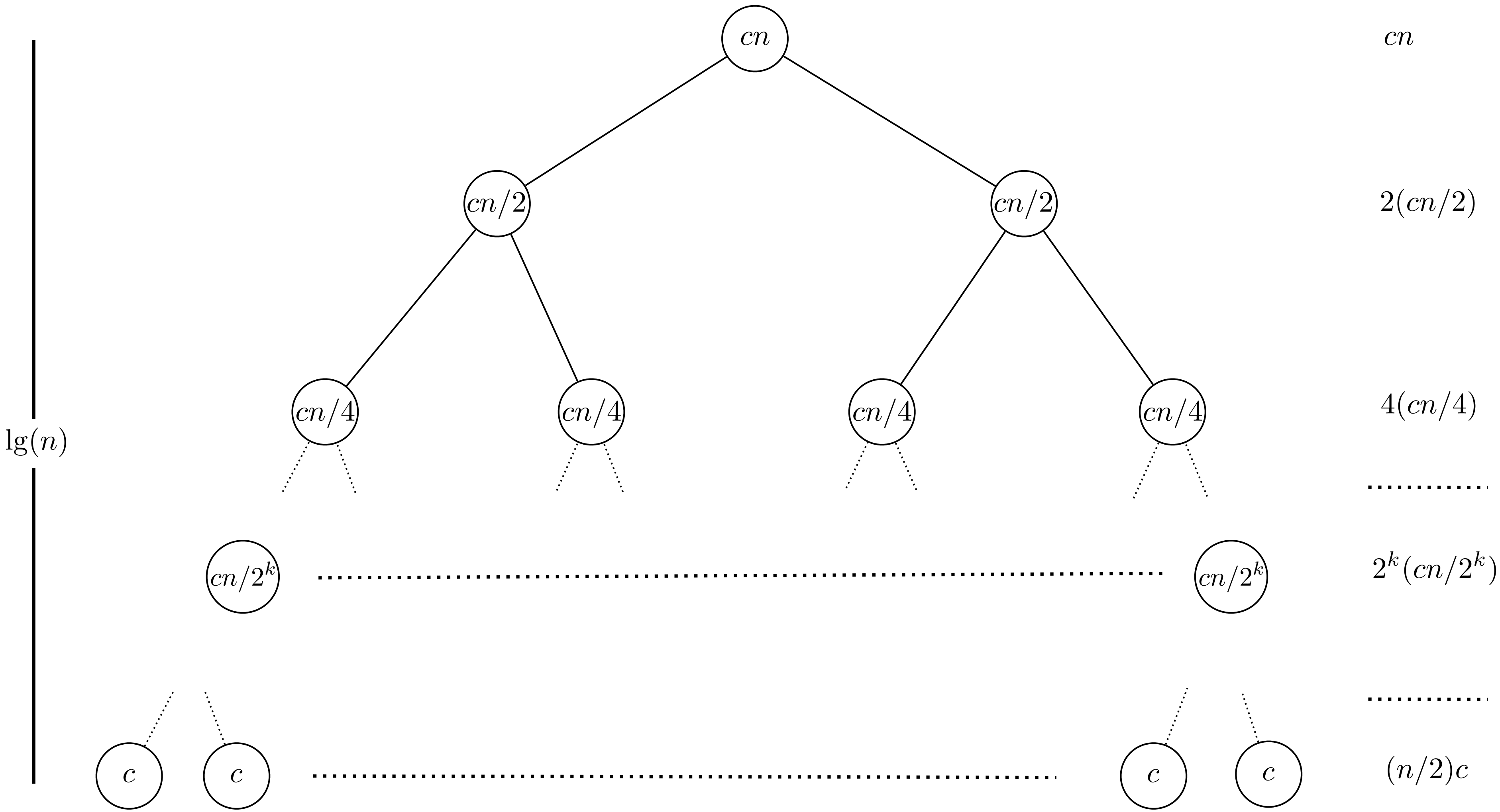

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More Recurrence Relations

More recurrences

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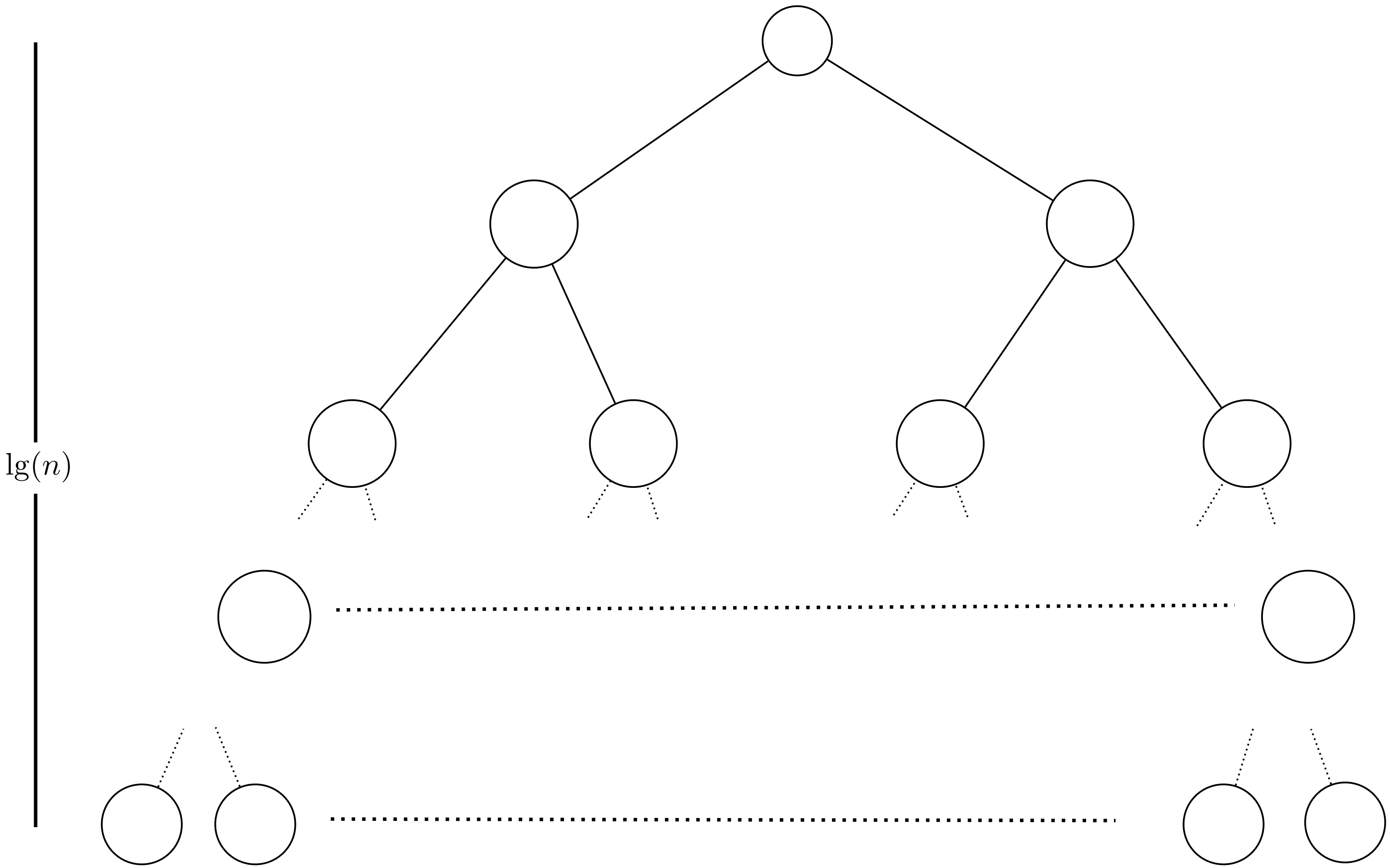

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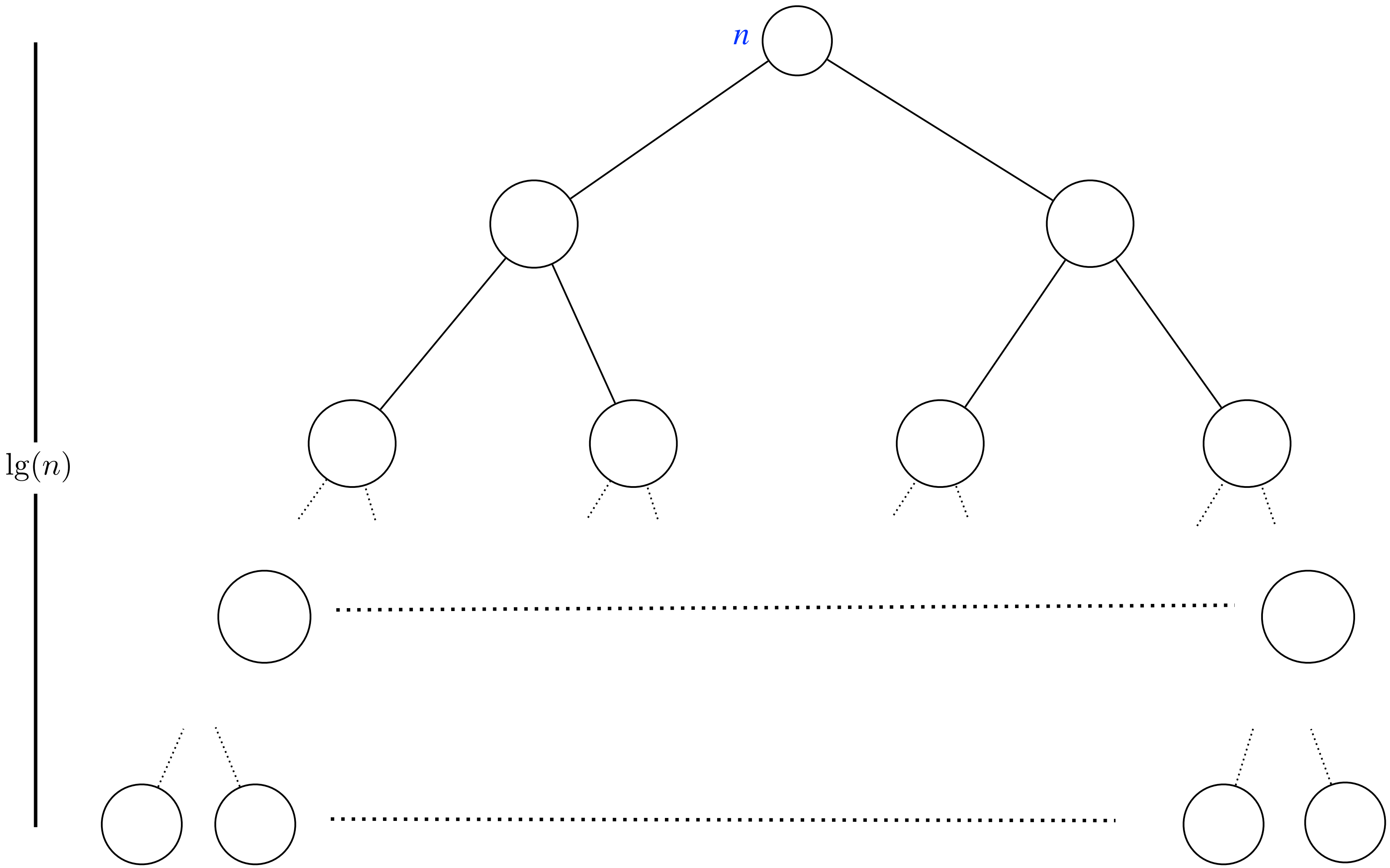

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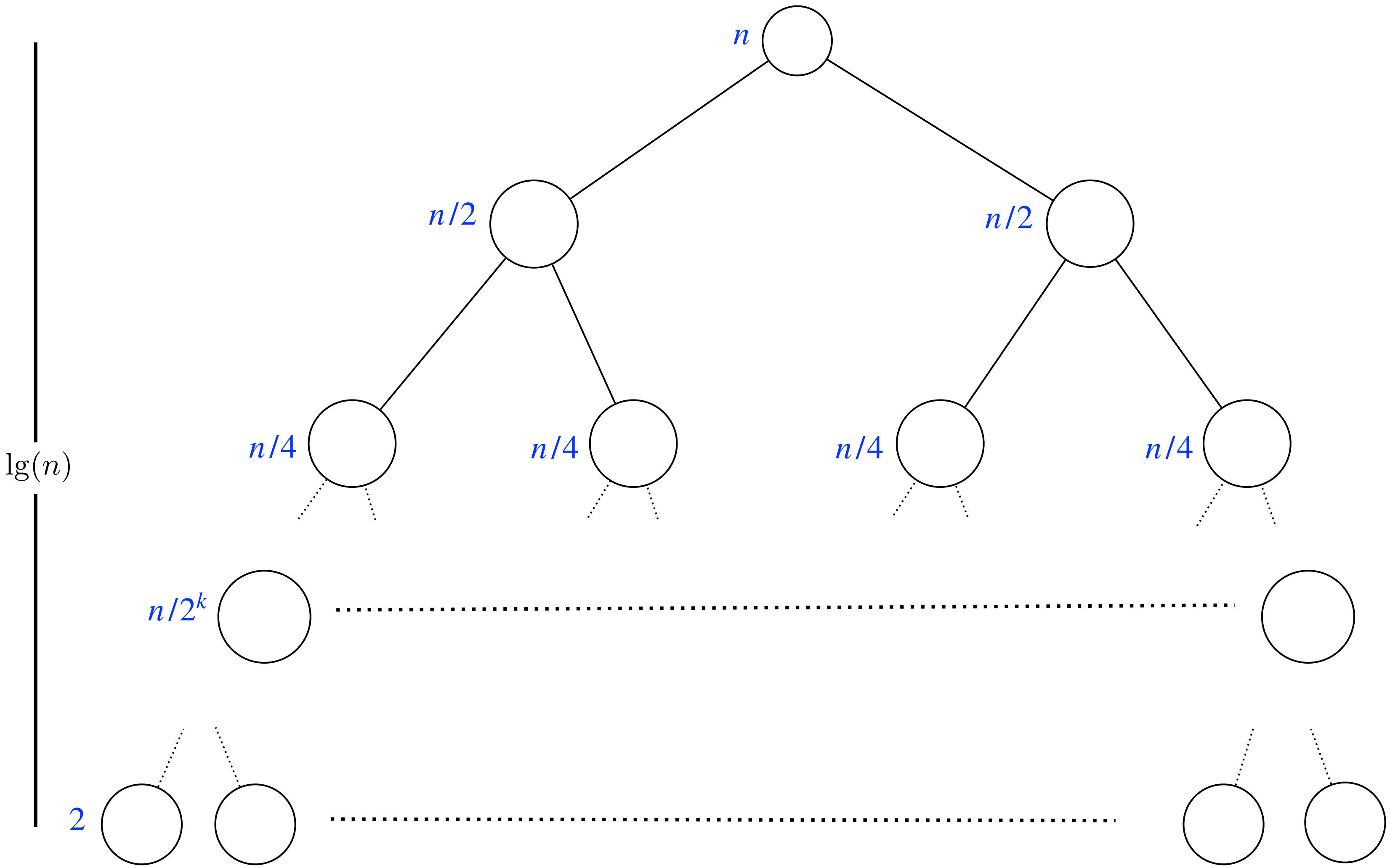

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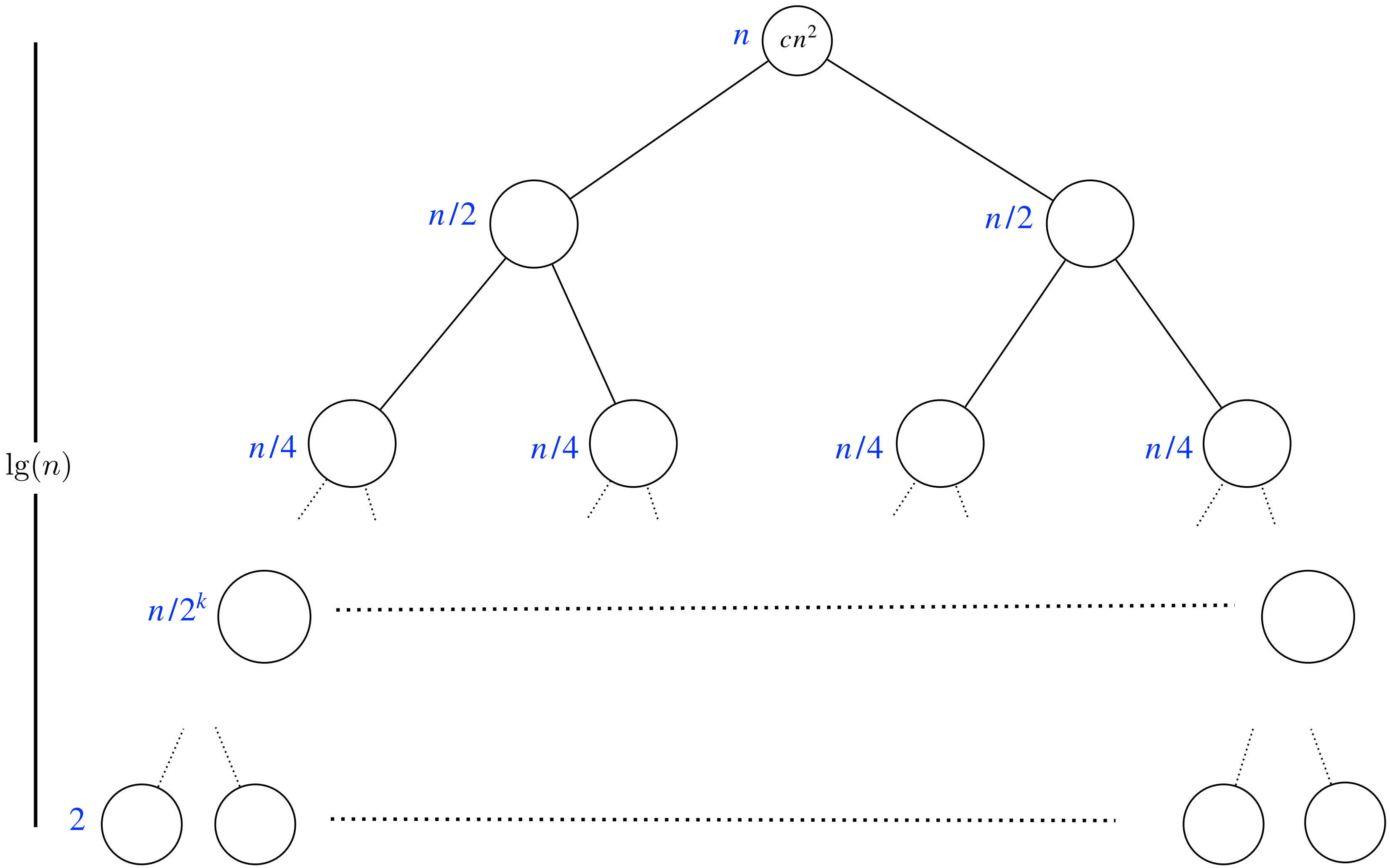

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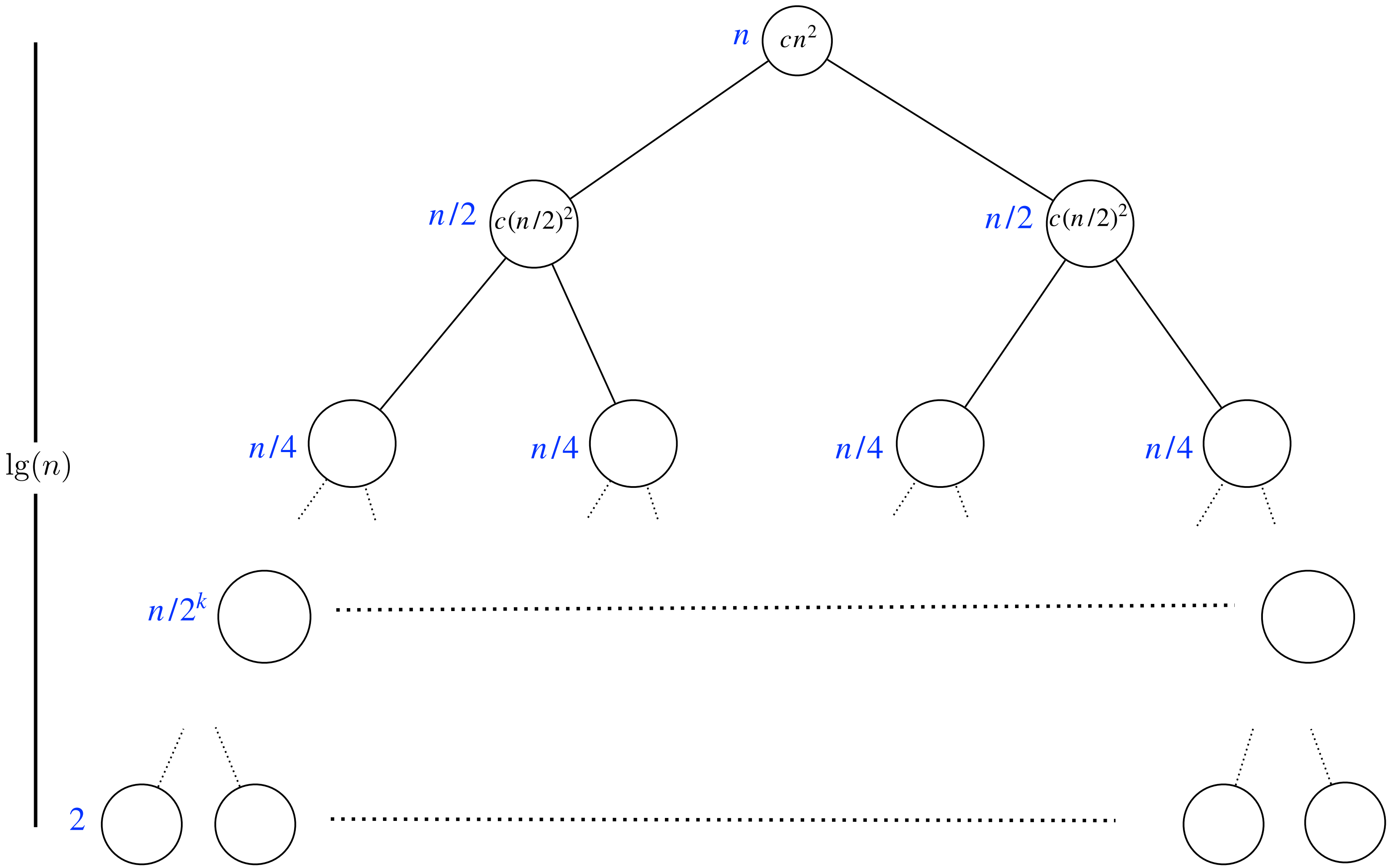

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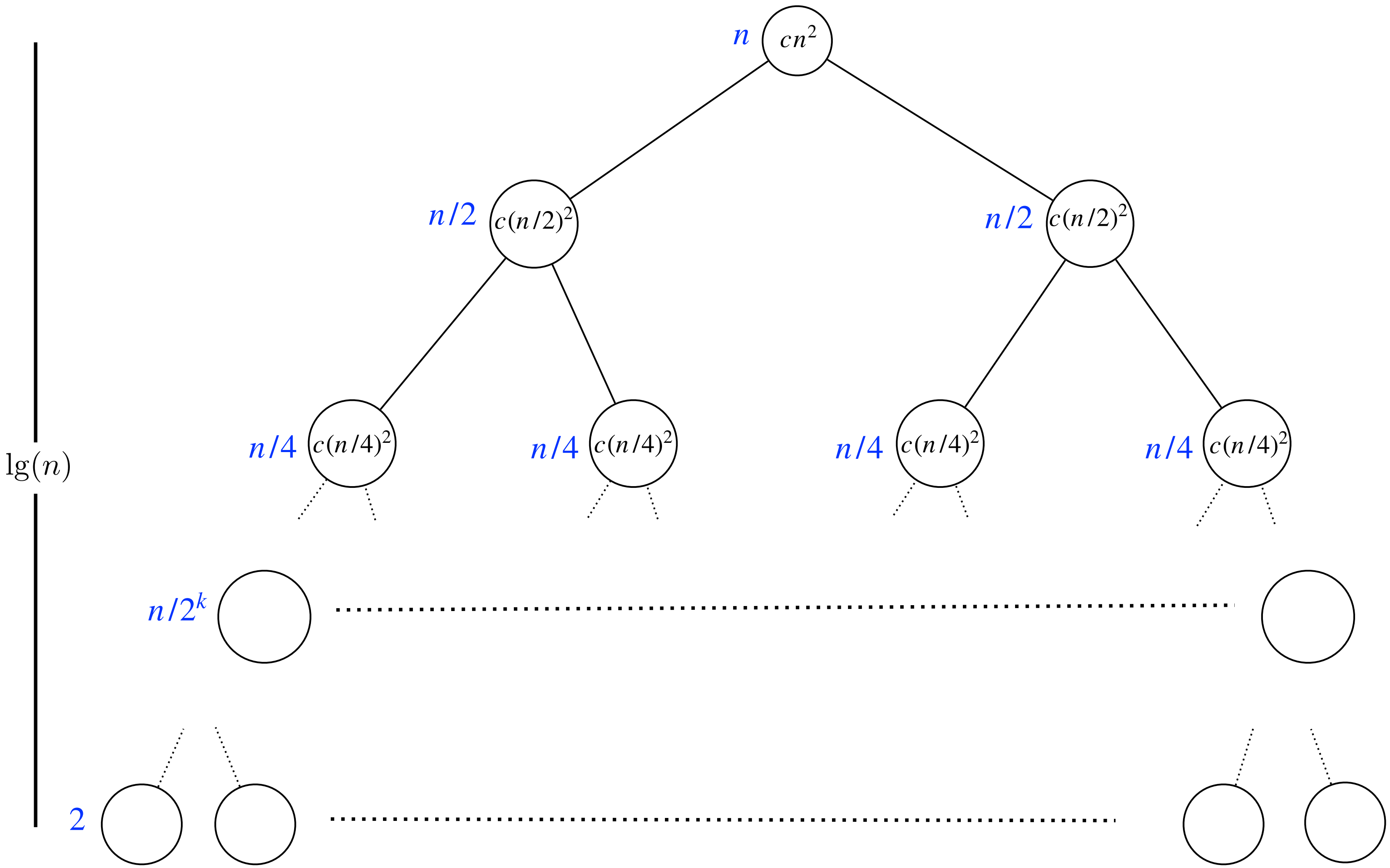

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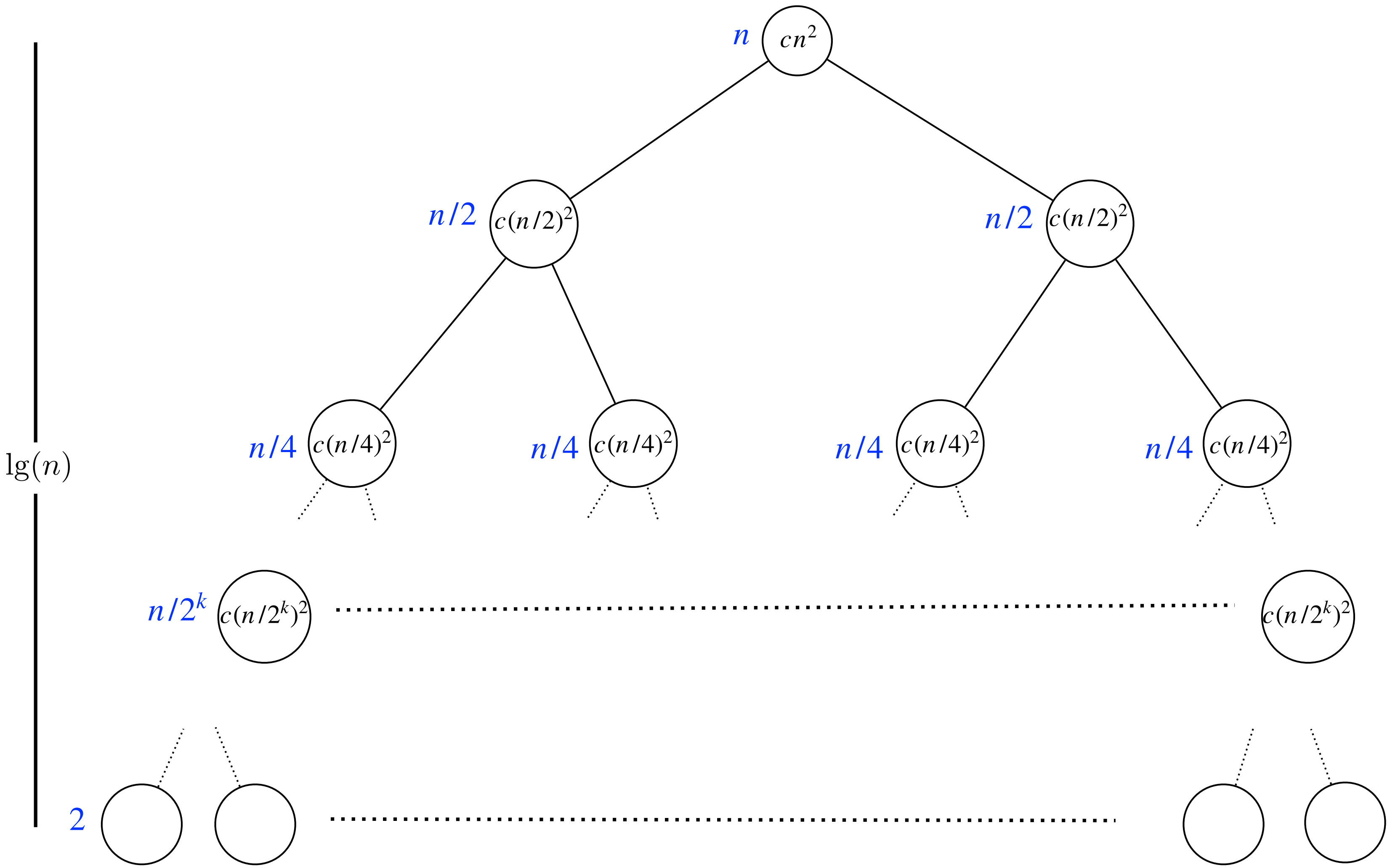

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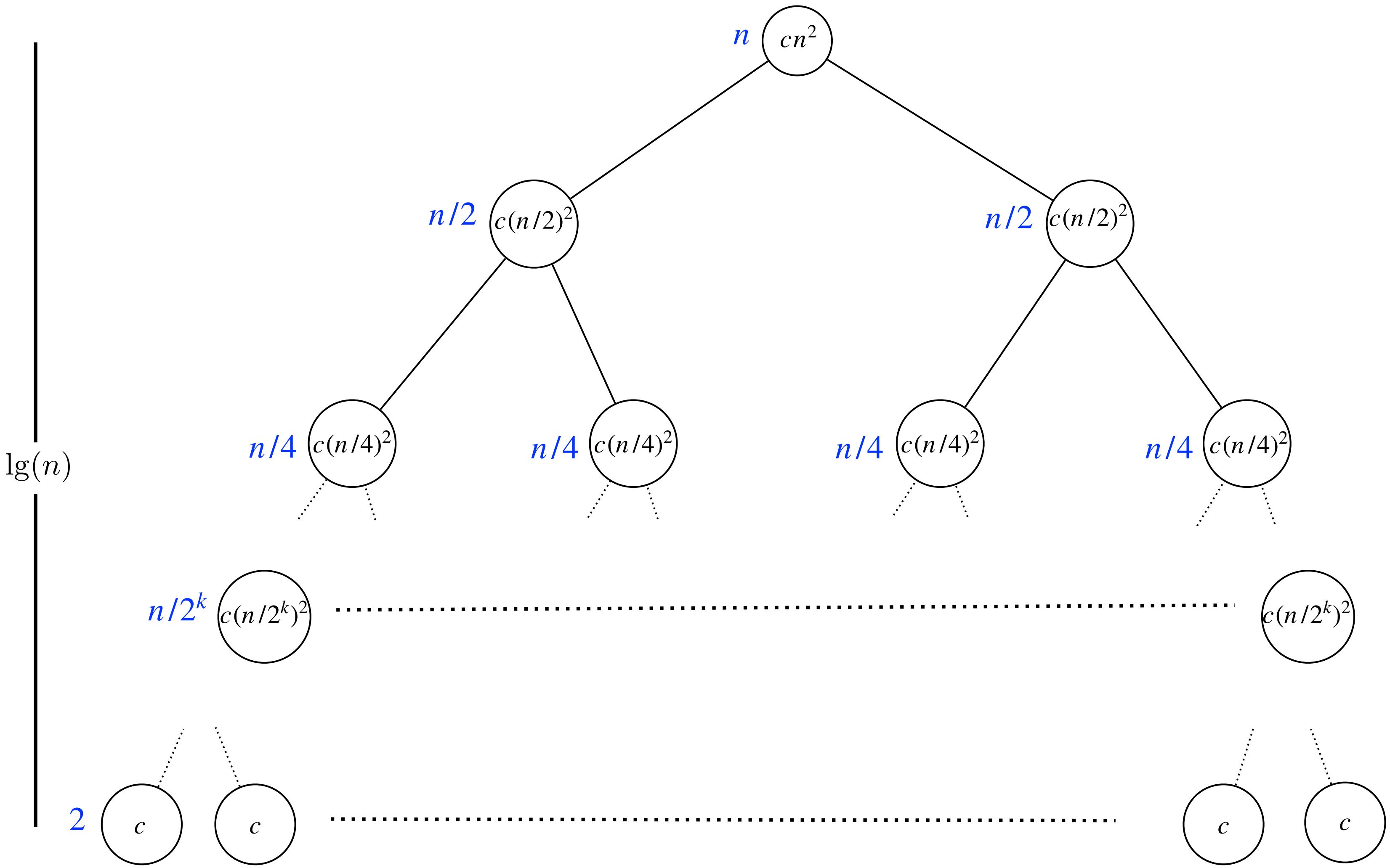

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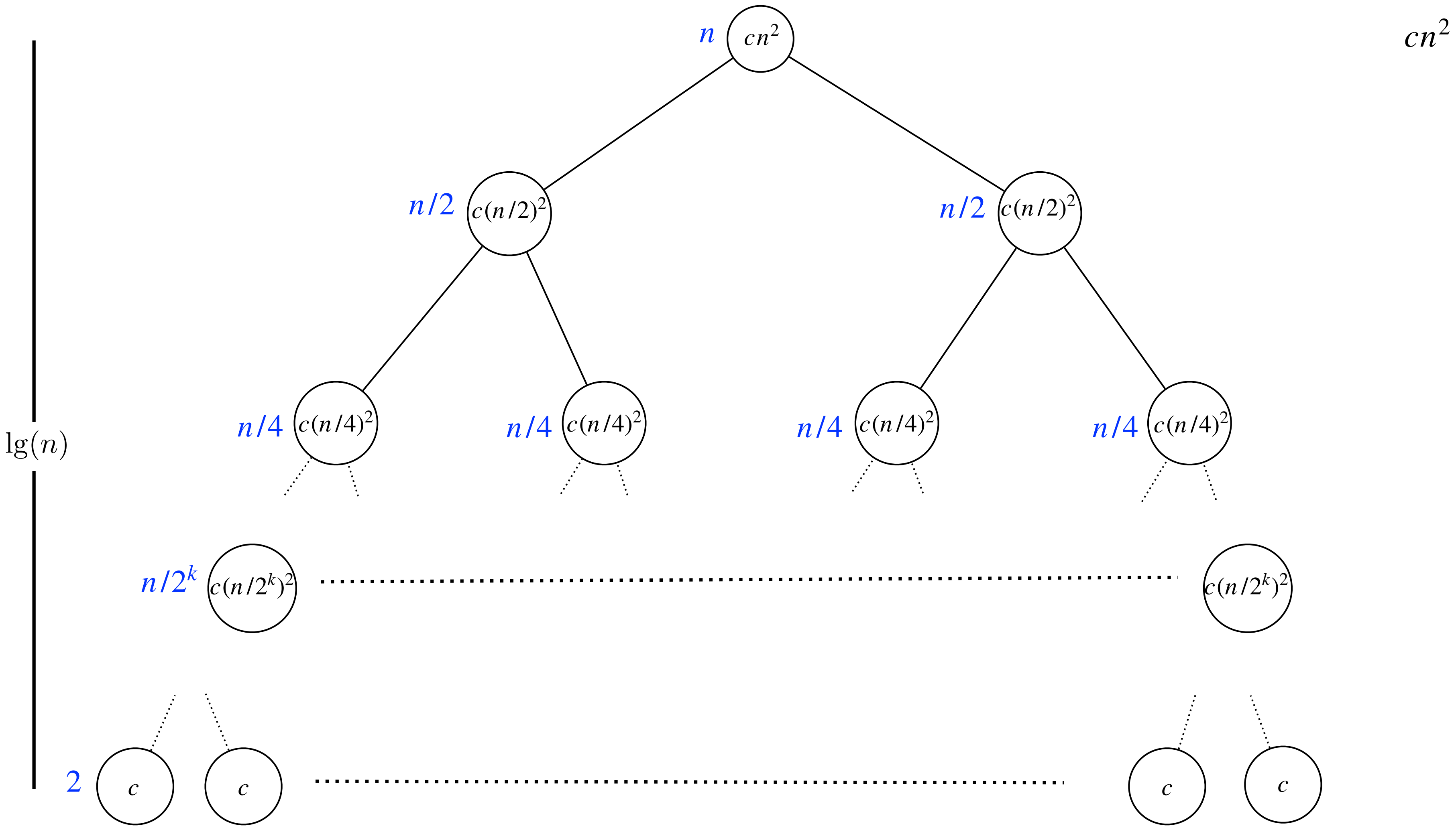

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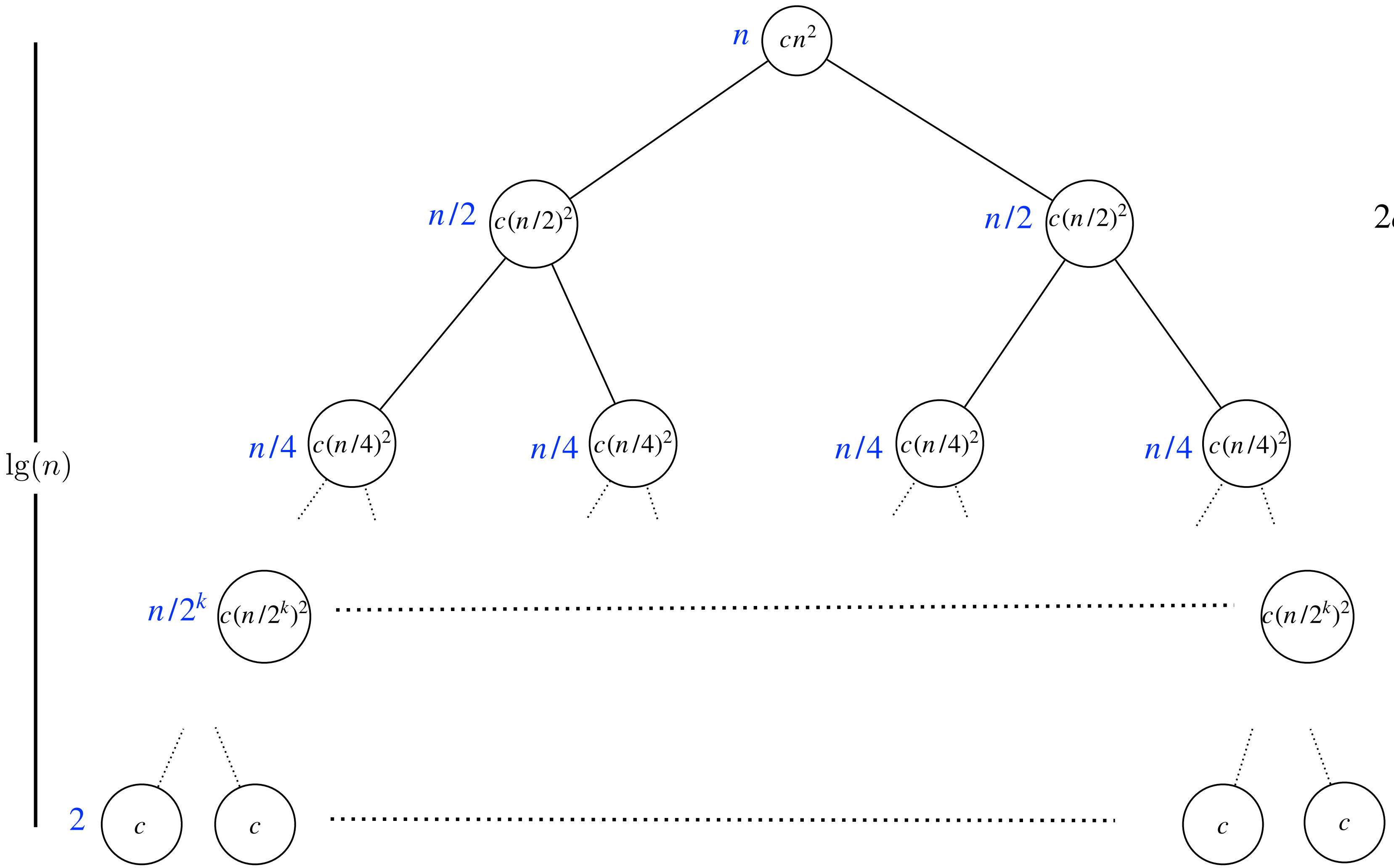

cn^2

$2c(n/2)^2 = \dfrac{cn^2}{2}$

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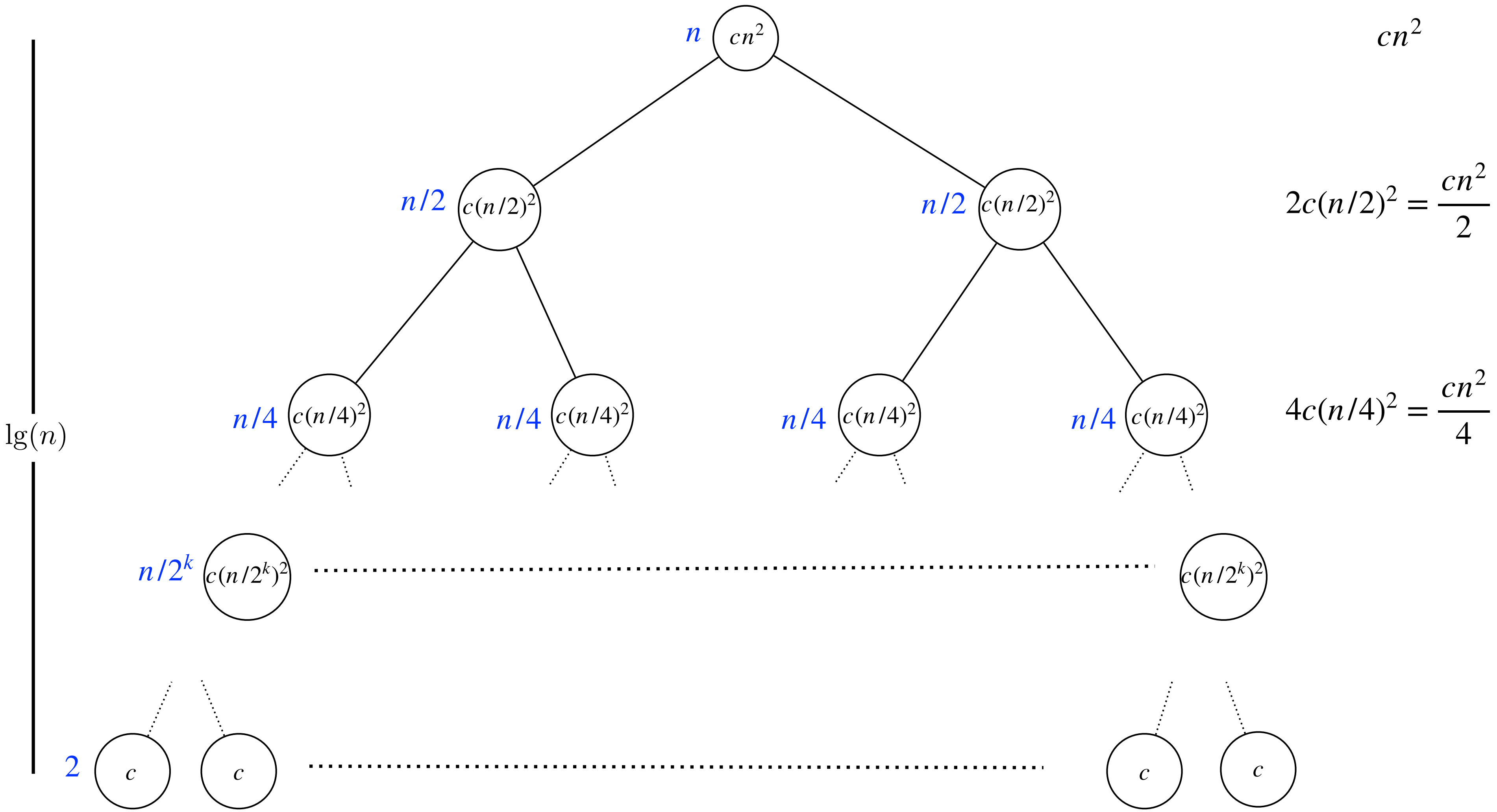

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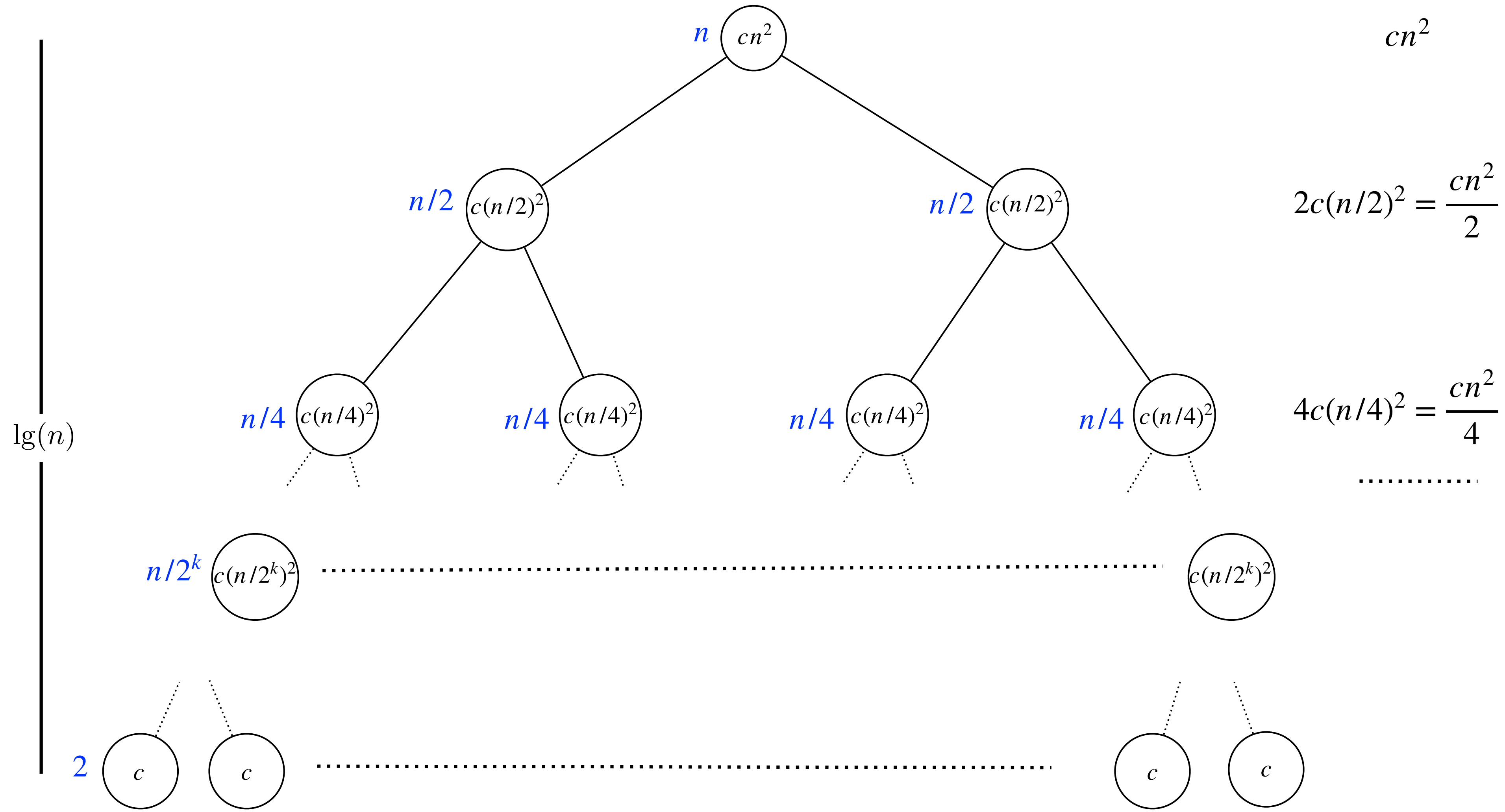

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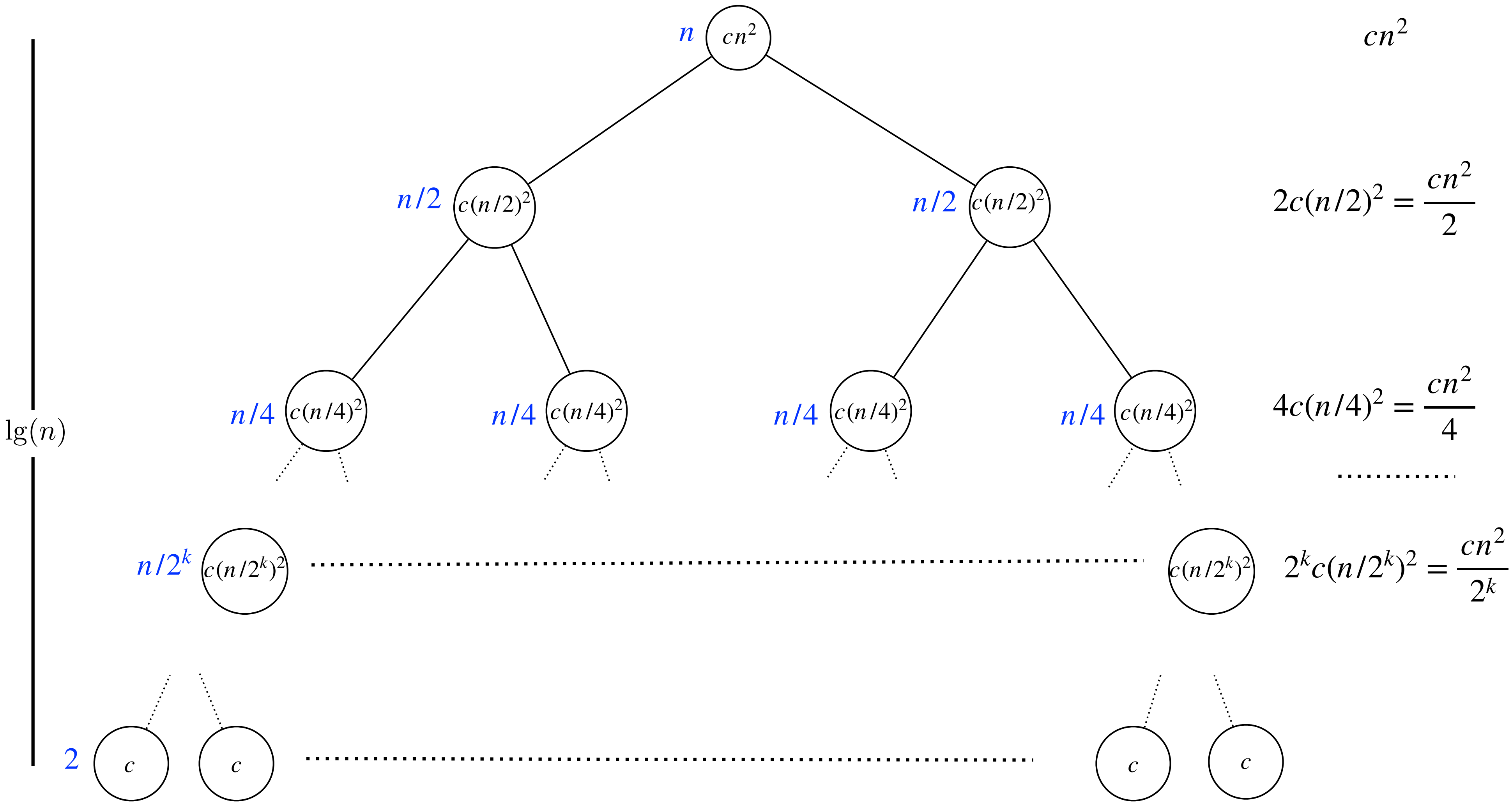

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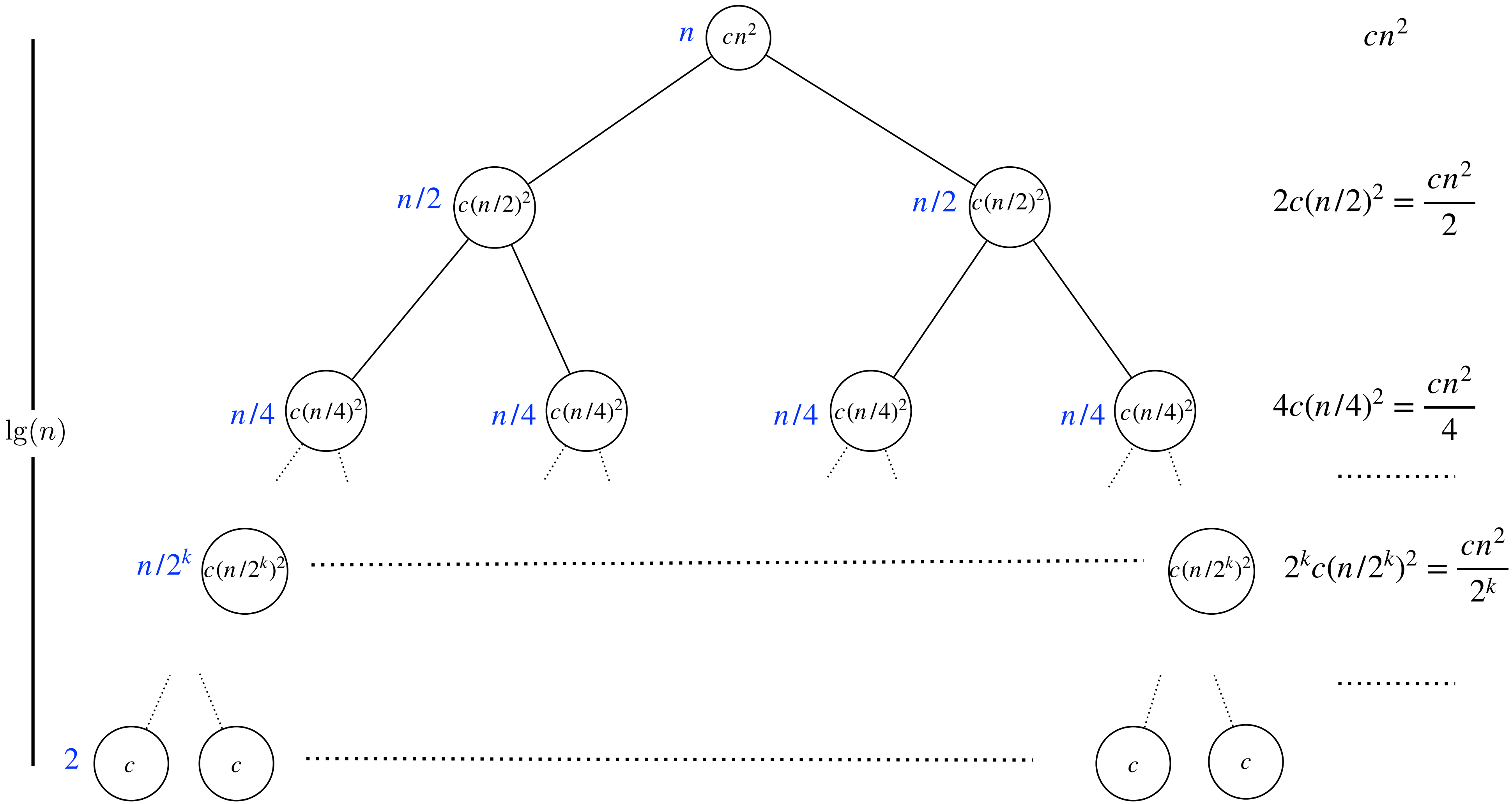

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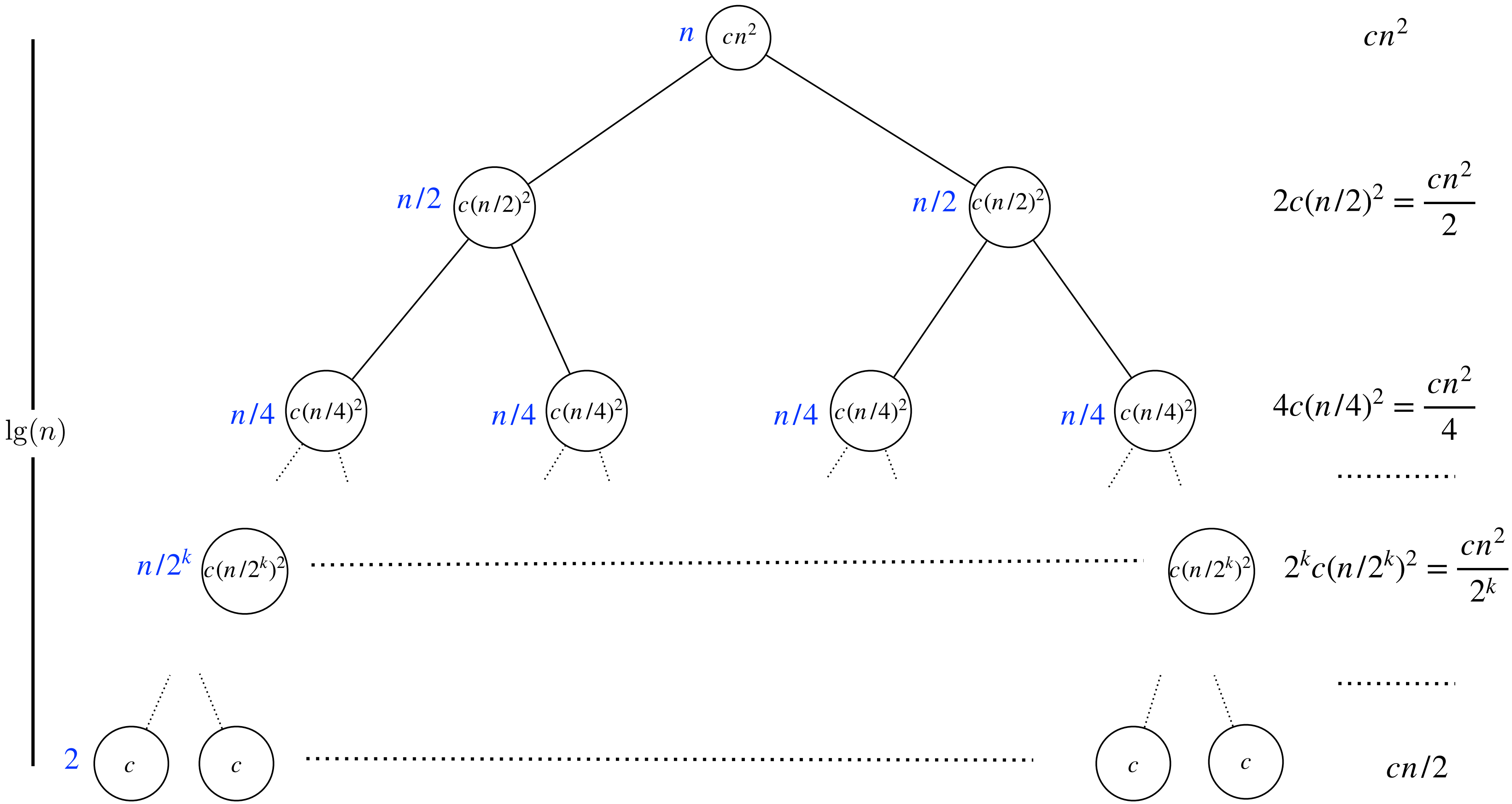

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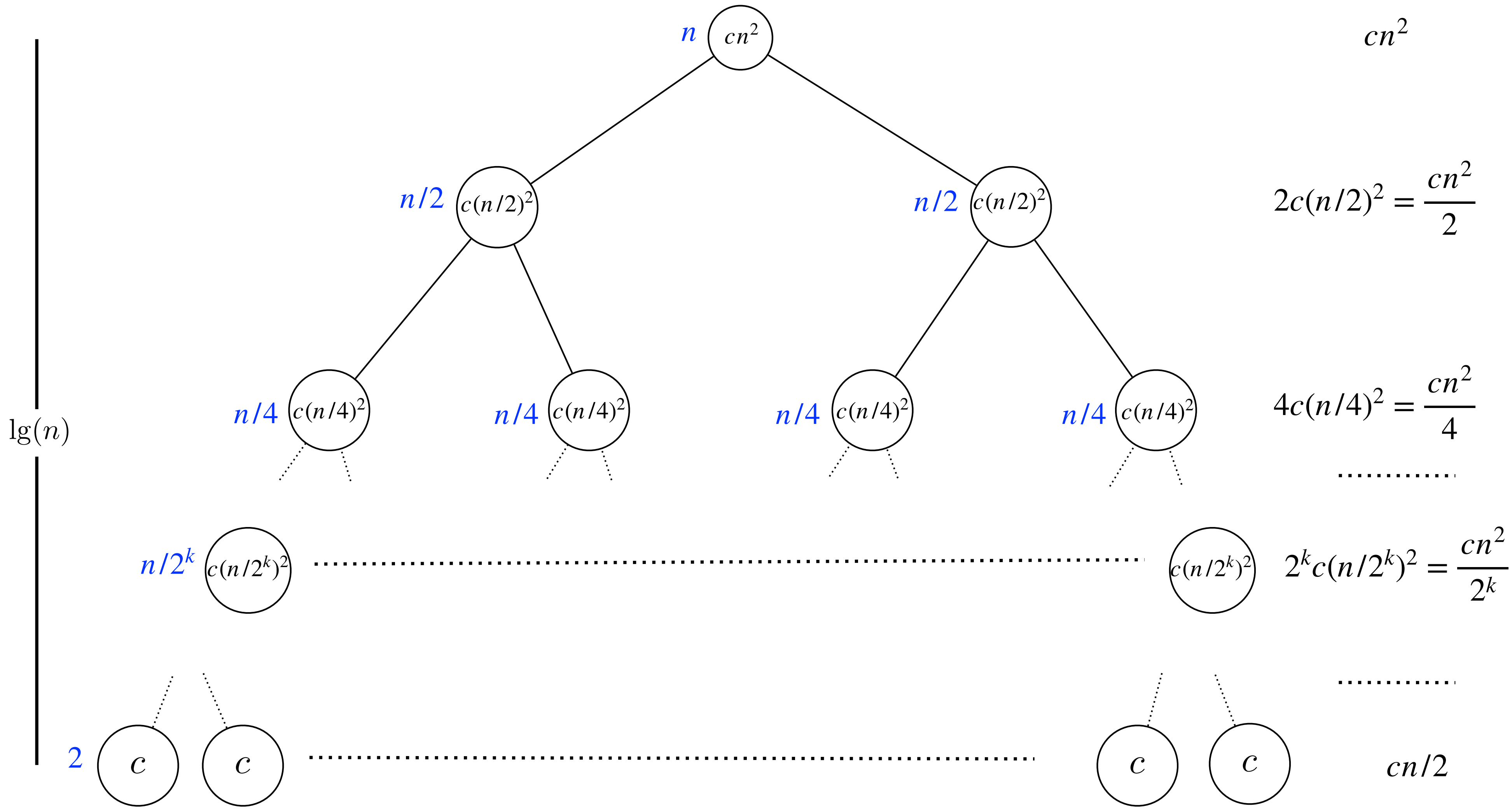

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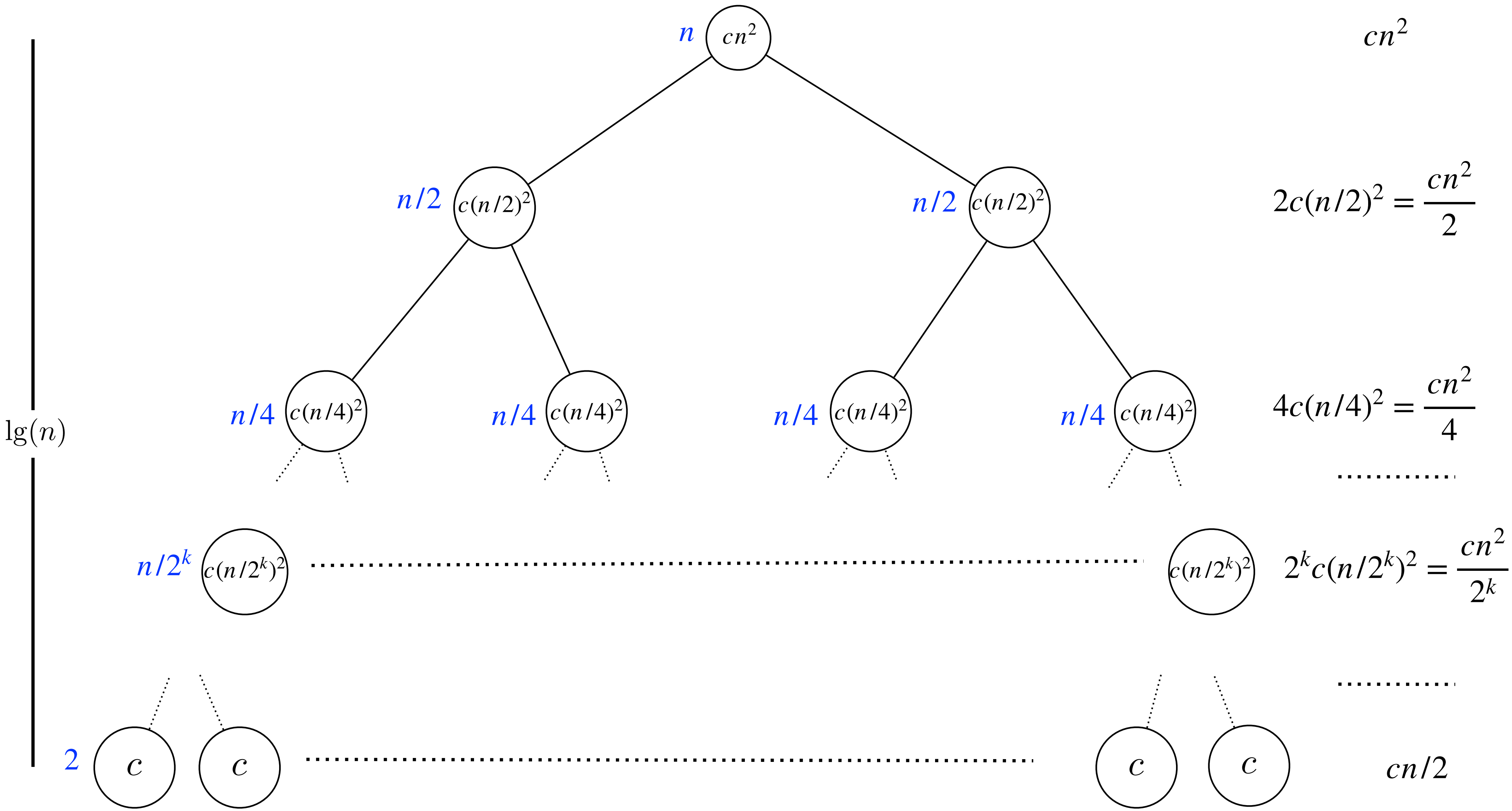

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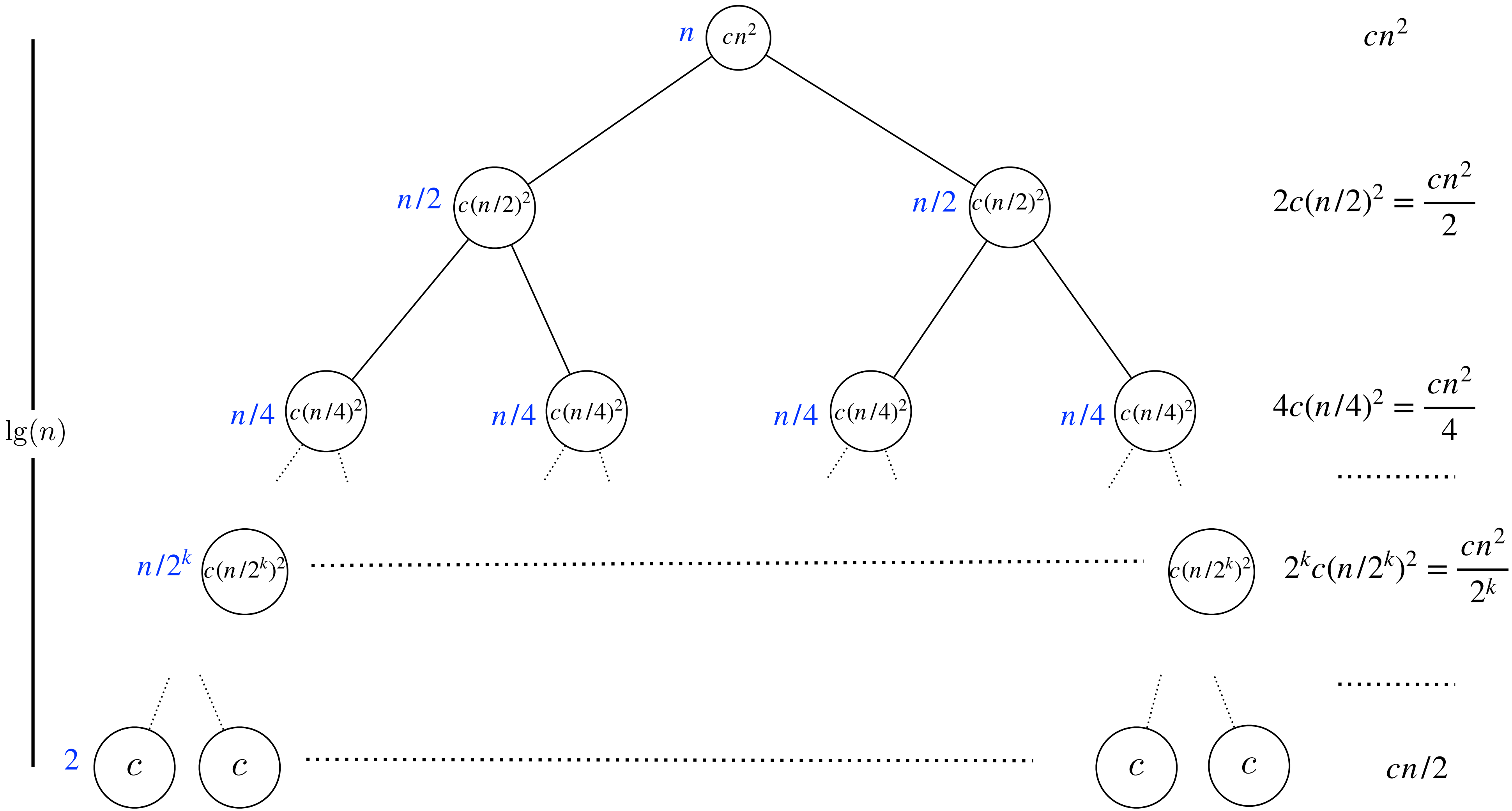

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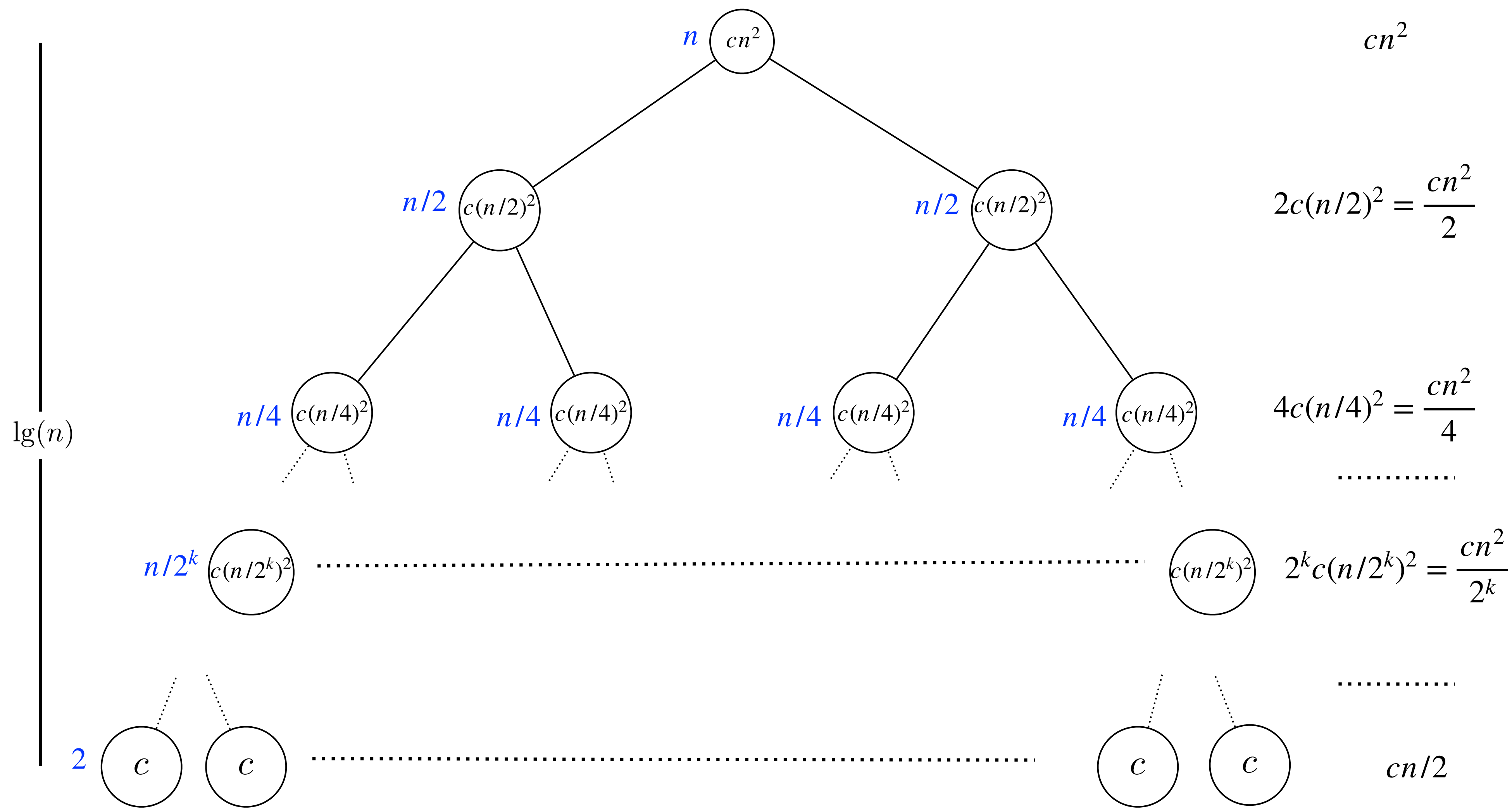

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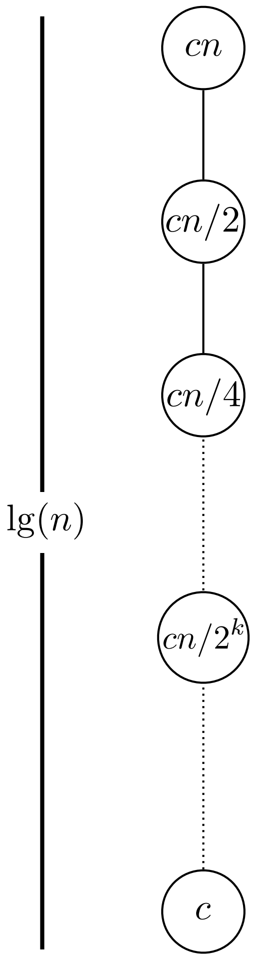

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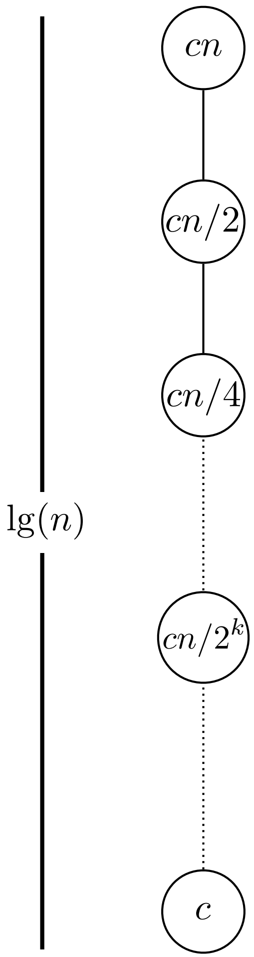

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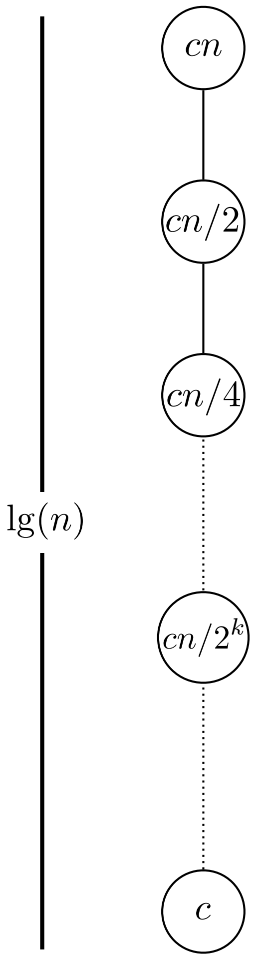

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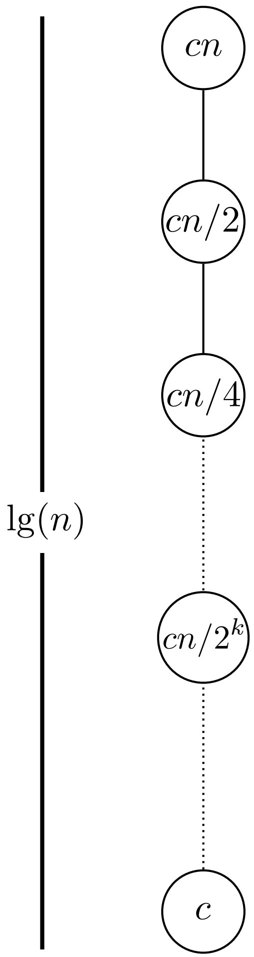

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- Substitution: Guess $T(n) \leq kn$

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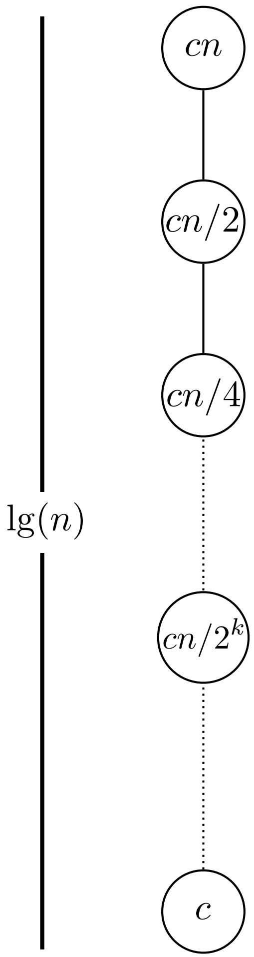

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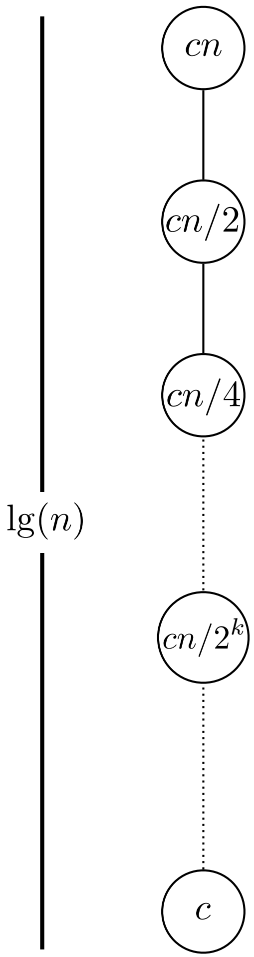

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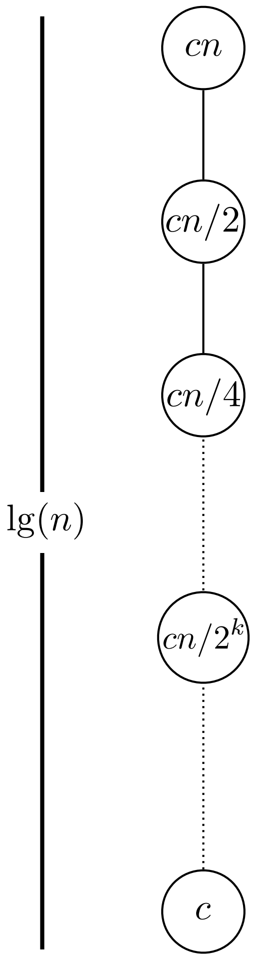

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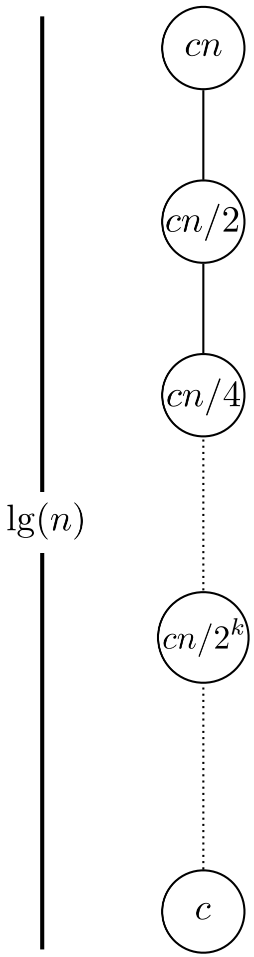

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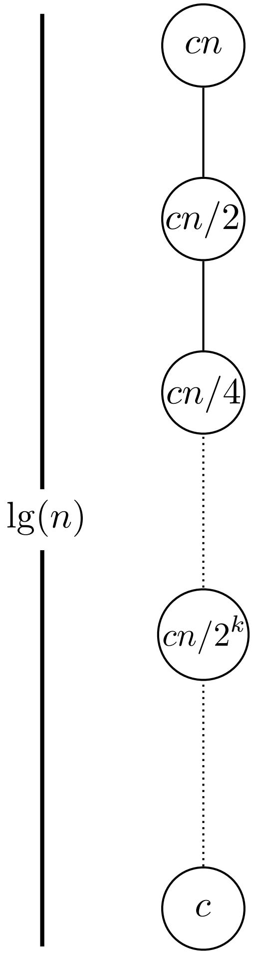

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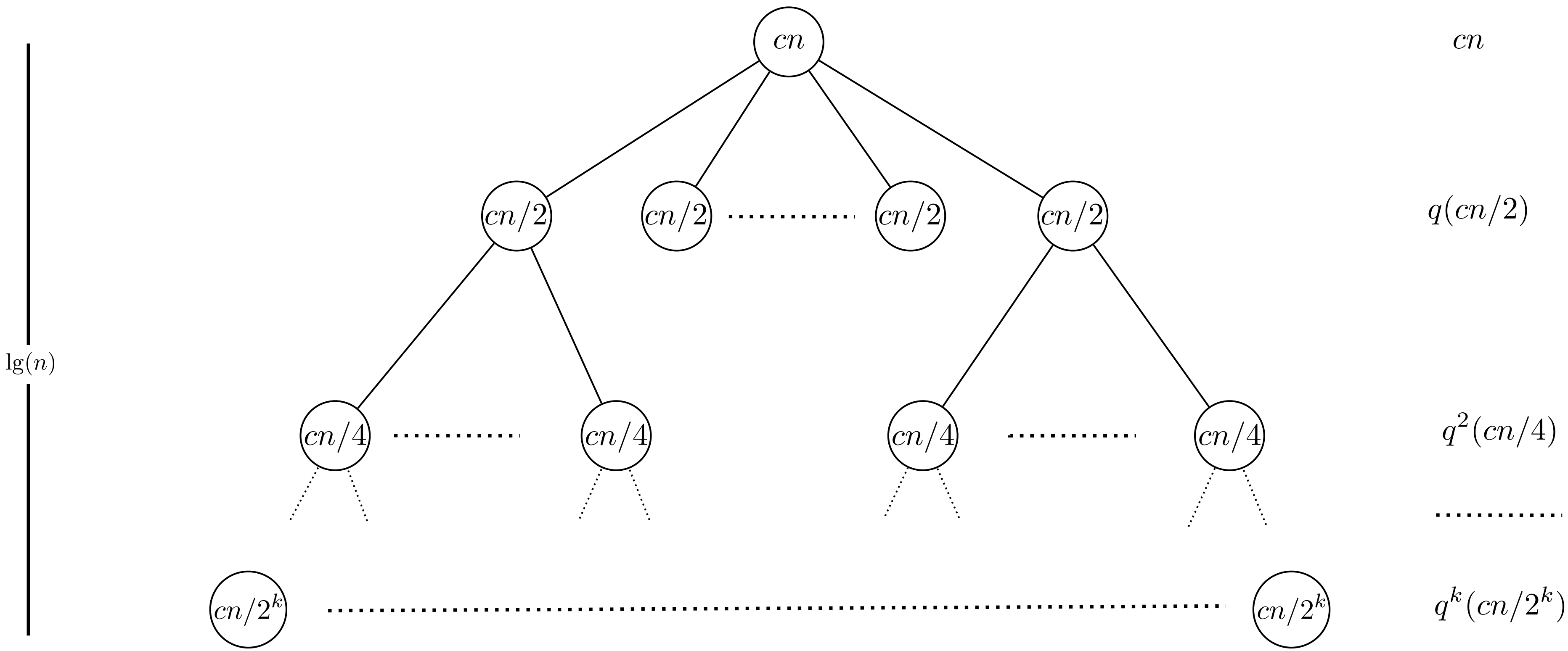

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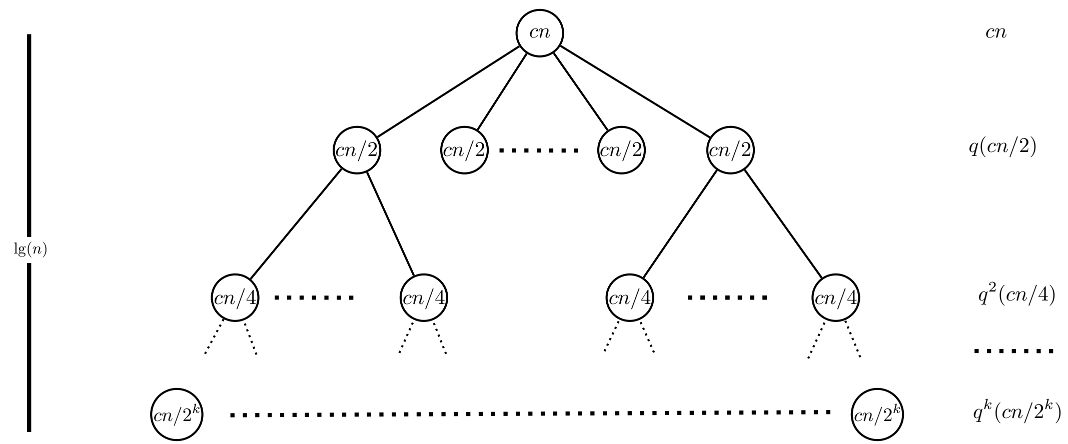

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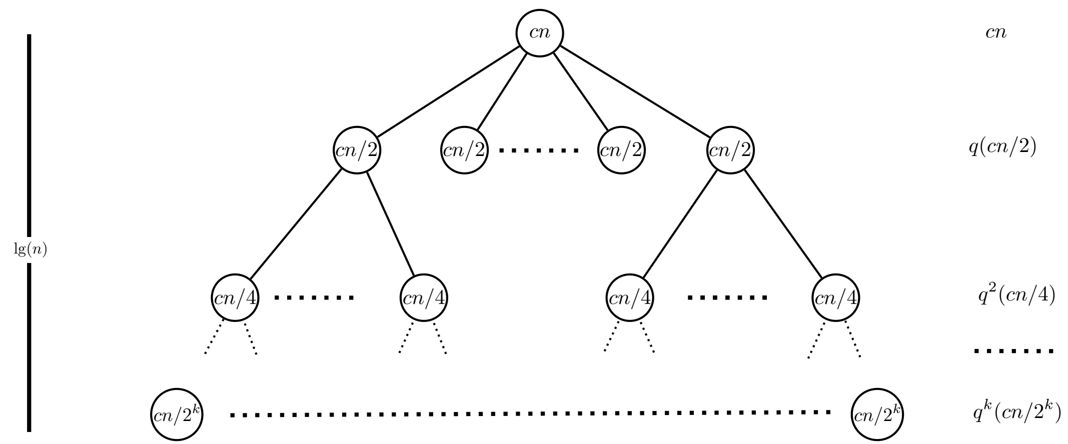

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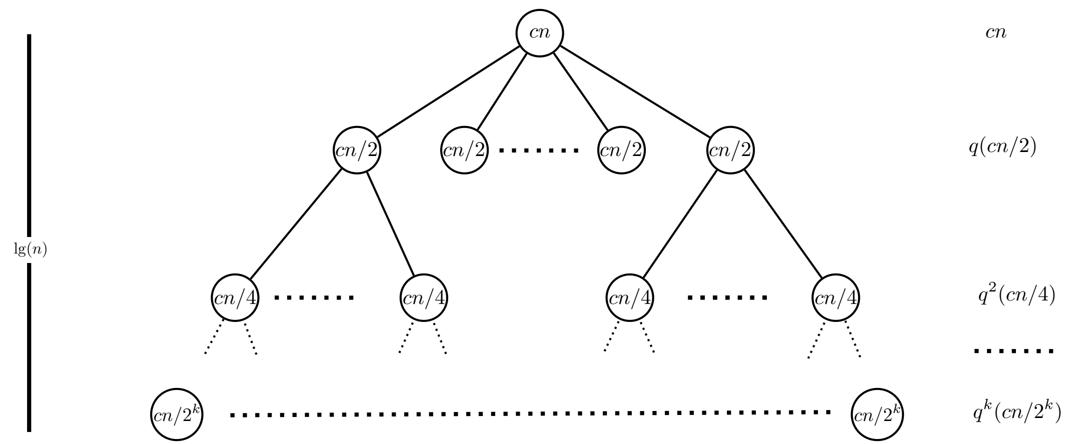

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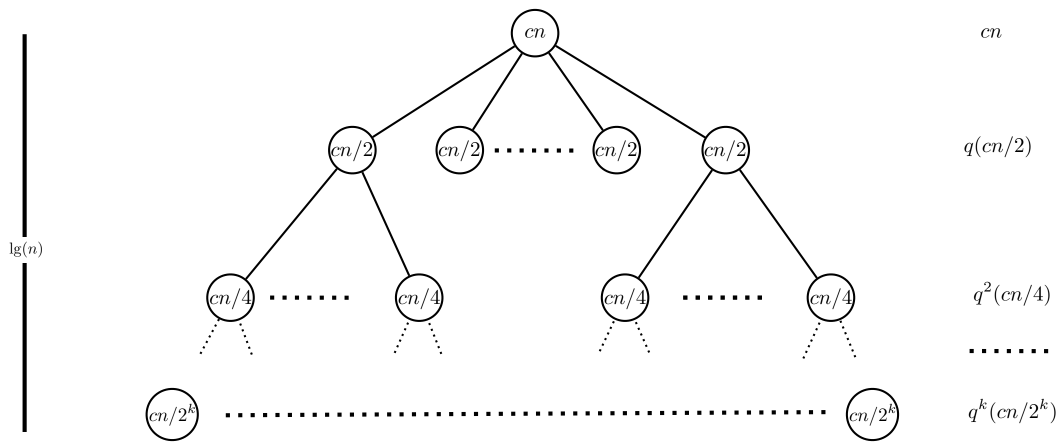

Geometric series.

for $x \neq 1 : \displaystyle\sum_{i=0}^{m} x^i = \frac{x^{m+1} - 1}{x - 1}$

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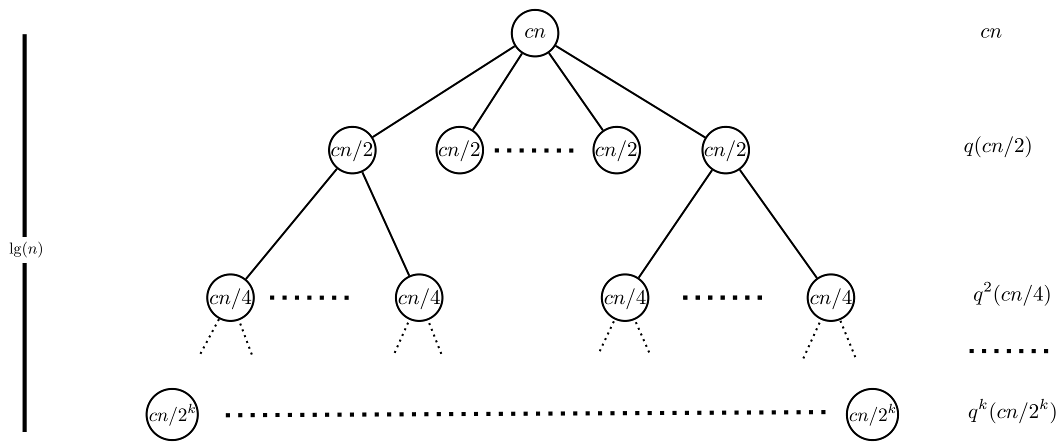

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Integer multiplication

- Add. Given two n-bit integers a and b, compute a + b.

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- School method. $\Theta(n)$ bit operations.

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1	1	0	×	1	1	1
				0	0	0
+			1	1	1	0
+		1	1	1	0	0
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Integer multiplication: warmup

- Divide-and-conquer: divide the n-bit integers into two.

$$x = 10001101 \qquad y = 11100001$$

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Integer multiplication: Karatsuba

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- Karatsuba:
  - Recursively compute *three* products of n/2-bit integers:

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1 2 1 3 3

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- Karatsuba:
  - Recursively compute *three* products of n/2-bit integers:
    - $x_1 y_1, (x_1 + x_0)(y_1 + y_0), x_0 y_0$
  - Shift, add, and subtract to obtain result.

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$$T(n) = \underbrace{3T(n/2)}_{\text{recursive calls}} + \underbrace{cn}_{\text{add, shift}}$$

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  - Recursively compute *three* products of n/2-bit integers:
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  - Shift, add, and subtract to obtain result.

$$\begin{gathered} (x_1 + x_0)(y_1 + y_0) = \\ x_1 y_1 + x_1 y_0 + x_0 y_1 + x_0 y_0 \\ \Rightarrow \\ x_1 y_0 + x_0 y_1 = \\ (x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0 \end{gathered}$$

$$T(n) \;=\; \underbrace{3T(n/2)}_{\text{recursive calls}} \;+\; \underbrace{cn}_{\text{add, shift}}$$

$$T(n) = O(n^{\lg 3}) = O(n^{1.59})$$