

# Technical University of Denmark

Written exam, December 11, 2015.

Course name: Algorithms and data structures.

Course number: 02110.

Aids allowed: All written materials are permitted.

Exam duration: 4 hours

Weighting: Question 1: 6% - Question 2: 24% - Question 3: 15% - Question 4: 15% - Question 5: 11% - Question 6: 18% - Question 7: 11%.

The weighting is only an approximative weighting.

You can answer the exam in either Danish or English.

**All questions should be answered by filling out the blank space below the question. As exam paper just hand in this and the following pages filled out. If you need more space you can use extra paper that you hand in together with the exam paper.**

## Question 1

In the following two questions, there might be more than one statement that is true (there might also be 0).

**Question 1.1** In which of the following data structures does an insertion take worst case  $O(\log n)$  time:

- A Red-black tree       B Splay tree       C 2-3-4 tree

**Question 1.2** In which of the following data structures does  $n$  insertions take worst case  $O(n \log n)$  time:

- A Red-black tree       B Splay tree       C 2-3-4 tree

**Question 1.3** Let  $T$  be a red-black tree and let  $S$  be a splay tree both with  $n$  nodes. Let  $d_T(x)$  be the depth of node  $x$  in  $T$  and let  $d_S(x)$  be the depth of node  $x$  in  $S$ . Let  $\text{insert}(x)$  be the last operation performed in both trees. Then:

- A  $d_T(x) \leq d_S(x)$        B  $d_S(x) \leq d_T(x)$        C  $d_T(x) = O(\log n)$        D  $d_S(x) = O(\log n)$

## Question 2

**Question 2.1** Draw the suffix tree for the string `pinepie$` (don't replace the labels by indexes into the string, just write the labels in the vertices):

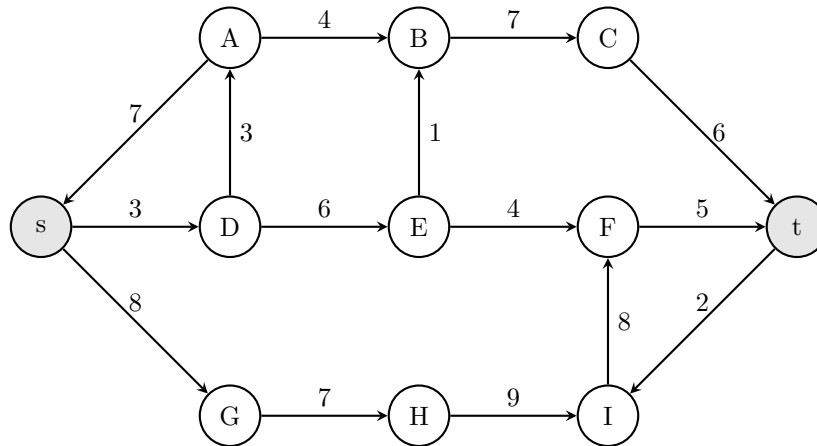
**Question 2.2** Compute the prefix function as used in the Knuth-Morris-Pratt algorithm for the string `pinepie`:

$i$	1	2	3	4	5	6	7
$P[i]$	$p$	$i$	$n$	$e$	$p$	$i$	$e$
$\pi[i]$							

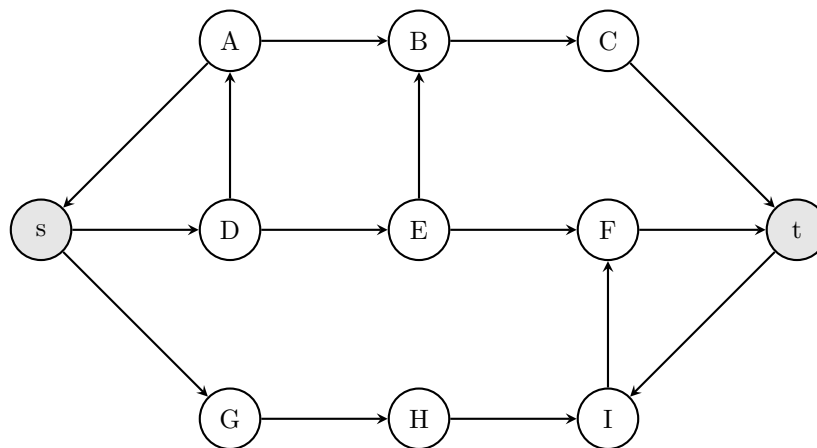
**Question 2.3** Draw the string matching finite automaton for the string `pinepie`:

### Question 3

Consider the network  $N$  below with capacities on the edges.



**Question 3.1** Give a maximum flow from  $s$  to  $t$  in the network  $N$  (write the flow for each edge along the edges on the graph below), give the value of the flow, and give a minimum  $s - t$  cut (give the partition of the vertices).



value of flow: \_\_\_\_\_

minimum cut: \_\_\_\_\_

**Question 3.2** Use Edmonds-Karp's algorithm to compute a maximum flow in the network  $N$ . For each augmenting path write the nodes on the path and the value you augment the path with in the table below.

augmenting path	value

## Question 4

The Dean has asked you to arrange the annual Christmas party for the students at DTU. You have to make a plan for how to place the tables in the hall. The local fire department has divided the hall up into an  $n \times m$  grid of subsquares and declared that you can place at most two tables in each row and at most one in each column. Unfortunately, the Dean who loves Christmas has put up Christmas trees in many of the subsquares. You cannot place a table in a subsquare with a Christmas tree.

**Example** Here  $n = 4$  and  $m = 8$ . The  $*$  are Christmas trees and  $T$  are tables. In the example the maximum number of tables that can be placed is 7.

*	$T$						$T$
$T$	*	*	$T$	*	*		*
	*	*		*	*	$T$	*
	*	$T$	*		$T$	*	

**Question 4.1** Model the problem as a graph problem. Explain how you model the problem as a graph problem and draw the graph corresponding to the example above.

**Question 4.2** Describe an algorithm that given  $n$ ,  $m$ , and the placement of the Christmas trees computes the maximum number of tables you can place in the hall. Analyze the asymptotic running time of your algorithm. Remember to argue that your algorithm is correct.



## Question 5

You are putting together a set of Christmas songs that will be handed out at the party. The Dean has declared that every song must contain the sentence "Merry\_Christmas\_Dear\_Dean", where "\_" denotes a blank space. E.g. the song:

```
We_wish_you_a_Merry_Christmas_  
We_wish_you_a_Merry_Christmas_  
We_wish_you_a_Merry_Christmas_  
Dear_Dean_  
Dear_Dean
```

contains one occurrence of of the sentence "Merry\_Christmas\_Dear\_Dean" (line breaks are disregarded).

Formally, you are given a set  $S$  of songs  $S_1, \dots, S_k$  and a sentence  $P$ . Song  $S_i$  contains  $n_i$  characters and  $P$  contains  $m$  characters. Let  $n = \sum_{i=1}^k n_i$  denote the total number of characters in the songs. All the strings are over an alphabet of size  $O(1)$ . Describe an algorithm that returns all the songs that contain  $P$ . Analyze the asymptotic running time of your algorithm. Remember to argue that your algorithm is correct.

## Question 6

At the Christmas party there will be a game called "Bing or Ding". There is a row of  $n$  red socks, each with an integer on it, which can be positive, negative, or zero. Let  $s_i$  denote the number on sock  $i$ . The rules are as follows:

- You have to consider all socks in order from sock 1 to  $n$ .
- At each sock you *must* say either "Bing!" or "Ding!".
- If you say "Bing!" at the  $i$ th sock you get  $s_i$  Christmas cookies.
- If you say "Ding!" at the  $i$ th sock you must pay  $s_i$  Christmas cookies.
- It is possible to end up with a negative number of Christmas cookies.
- It is forbidden to say the same word more than two times in a row. For example if you say "Bing!" at sock 5 and 6 you *must* say "Ding!" at sock 7.

Let  $L(1, i)$  denote the maximum number of Christmas cookies you can have after the first  $i$  socks if you say "Bing!" at sock  $i$ . Similarly, let  $L(0, i)$  denote the maximum number of Christmas cookies you can have after the first  $i$  if you say "Ding!" at sock  $i$ . Let  $L(b, 0) = 0$  for  $b \in \{0, 1\}$ .

**Question 6.1** Fill out the table below for  $L(b, i)$  when  $S = [s_1, s_2, s_3, s_4] = [3, 4, 8, -4]$ .

$L(b, i)$	0	1	2	3	4
0					
1					

**Question 6.2** Which of the following recurrences correctly computes  $L(b, i)$ :

$$\boxed{\text{A}} \quad L(b, i) = \begin{cases} 0 & \text{if } i = 0 \\ L(0, i - 1) + s_i & \text{if } b = 1 \\ L(1, i - 1) - s_i & \text{if } b = 0 \end{cases}$$

$$\boxed{\text{B}} \quad L(b, i) = \begin{cases} 0 & \text{if } i = 0 \\ s_1 & \text{if } b = 1 \text{ and } i = 1 \\ -s_1 & \text{if } b = 0 \text{ and } i = 1 \\ \max\{L(1, i - 2) - s_{i-1}, L(0, i - 2) + s_{i-1}\} + s_i & \text{if } b = 1 \text{ and } i \geq 2 \\ \max\{L(1, i - 2) + s_{i-1}, L(0, i - 2) - s_{i-1}\} - s_i & \text{if } b = 0 \text{ and } i \geq 2 \end{cases}$$

$$\boxed{\text{C}} \quad L(b, i) = \begin{cases} 0 & \text{if } i = 0 \\ s_1 & \text{if } b = 1 \text{ and } i = 1 \\ -s_1 & \text{if } b = 0 \text{ and } i = 1 \\ \max\{L(0, i - 1), L(0, i - 2) + s_{i-1}\} + s_i & \text{if } b = 1 \text{ and } i \geq 2 \\ \max\{L(1, i - 1), L(1, i - 2) - s_{i-1}\} - s_i & \text{if } b = 0 \text{ and } i \geq 2 \end{cases}$$

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**Question 6.3** Write pseudocode for an algorithm *based on dynamic programming and the recurrence from Question 6.2* that finds the maximum number of Christmas cookies you can end up with given  $n$  socks. Analyze the space usage and running time of your algorithm in terms of  $n$ .

## Question 7

During the party the Dean suddenly wants to know who won most Christmas cookies in the "Bing or Ding" game. He suggests the following algorithm:

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**Algorithm 1:** Find student with most cookies

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```
max ←  $-\infty$ 
s ← null
Randomly order the students. Let  $s_1, \dots, s_n$  be the students in this random order.
Let  $c_i$  denote the number of cookies won by student  $s_i$ .
for  $i = 1, \dots, n$  do
  if if  $c_i > \text{max}$  then
    | max ←  $c_i$  and  $s \leftarrow s_i$   (*)
  end
end
return  $s$ 
```

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In the following assume that all students won a different amount of cookies. That is,  $c_i \neq c_j$  for all  $i \neq j$ .

**Question 7.1** What is the probability that the line (\*) is executed at the last iteration?

**Question 7.2** Let  $X_i$  be a random variable that is 1 if line (\*) is executed in iteration  $i$  and 0 otherwise. What is the probability that  $X_i = 1$ ?

**Question 7.3** What is the expected number of times line (\*) is executed?