Dynamic Programming

Algorithm Design 6.1, 6.2, 6.4

Thank you to Kevin Wayne for inspiration to slides

Applications

- In class (today and next time)

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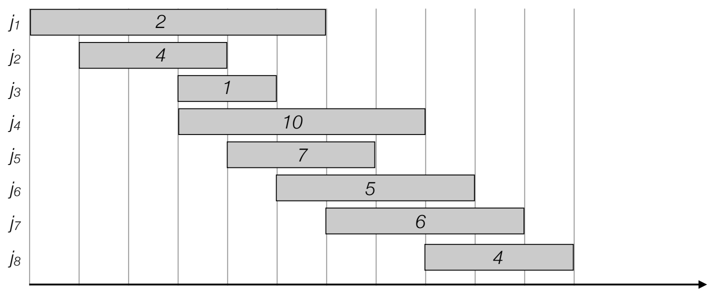
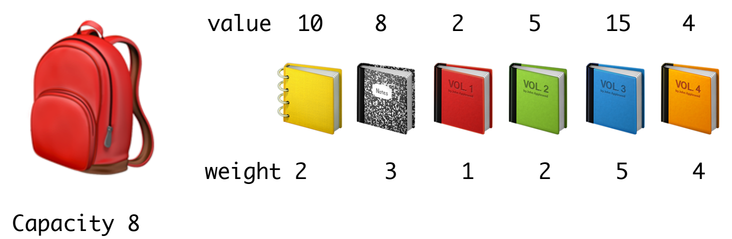
  - Weighted interval scheduling
    - Set of weighted intervals with start and finishing times
    - Goal: find maximum weight subset of non-overlapping intervals



Applications

- Today and next time
  - Weighted interval scheduling
  - Subset Sum and Knapsack
    - Set of items each having a weight and a value
    - Knapsack with a bounded capacity
    - Goal: fill knapsack so as to maximise the total value.



Applications

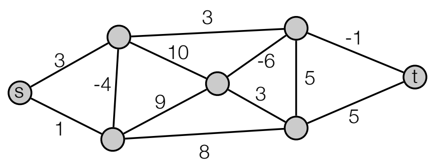
- Today and next time
  - Weighted interval scheduling
  - Subset Sum and Knapsack
  - Sequence alignment
    - Given two strings A and B how many edits (insertions, deletions, relabelings) is needed to turn A into B?

```
A C A A G T C          A C A A - G T C
- C A T G T -          - C A - T G T -

1 mismatch, 2 gaps      0 mismatches, 4 gaps
```


Applications

- Today and next time
  - Weighted interval scheduling
  - Subset Sum and Knapsack
  - Sequence alignment
  - Shortest paths with negative weights
    - Given a weighted graph, where edge weights can be negative, find the shortest path between two given vertices.



Applications

- Today and next time
  - Weighted interval scheduling
  - Subset Sum and Knapsack
  - Sequence alignment
  - Shortest paths with negative weights
- Some other famous applications
  - Unix diff for comparing 2 files
  - Vovke-Kasami-Younger for parsing context-free grammars
  - Viterbi for hidden Markov models
  - ....

Dynamic Programming

- Greedy. Build solution incrementally, optimizing some local criterion.
- Divide-and-conquer. Break up problem into independent subproblems, solve each subproblem, and combine to get solution to original problem.
- Dynamic programming. Break up problem into overlapping subproblems, and build up solutions to larger and larger subproblems.
  - Can be used when the problem have "optimal substructure":
    - *Solution can be constructed from optimal solutions to subproblems*
    - *Use dynamic programming when subproblems overlap.*

Computing Fibonacci numbers

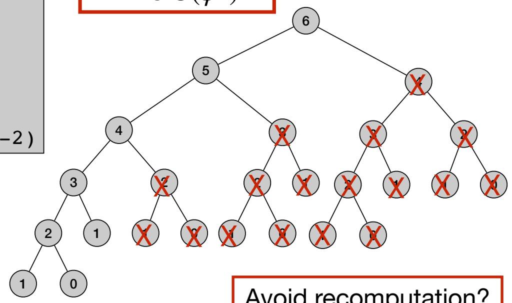
- Fibonacci numbers:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

- First try:

```
Fib(n)
if n = 0
  return 0
else if n = 1
  return 1
else
  return Fib(n-1) + Fib(n-2)
```

time $\Theta(\phi^n)$



Avoid recomputation?

Memoized Fibonacci numbers

- Fibonacci numbers:

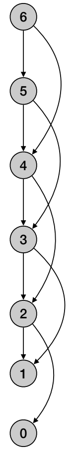
$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

- Remember already computed values:

```
for j=1 to n
  F[j] = null
Mem-Fib(n)

Mem-Fib(n)
if n = 0
  return 0
else if n = 1
  return 1
else
  if F[n] is empty
    F[n] = Mem-Fib(n-1) + Mem-Fib(n-2)
  return F[n]
```

time $\Theta(n)$



Bottom-up Fibonacci numbers

- Fibonacci numbers:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

- Remember already computed values:

```
Iter-Fib(n)
F[0] = 0
F[1] = 1
for i = 2 to n
  F[n] = F[n-1] + F[n-2]
return F[n]
```

time $\Theta(n)$

space $\Theta(n)$

Bottom-up Fibonacci numbers - save space

- Fibonacci numbers:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

- Remember last two computed values:

```
Iter-Fib(n)
previous = 0
current = 1
for i = 1 to n
  next = previous + current
  previous = current
  current = next
return current
```

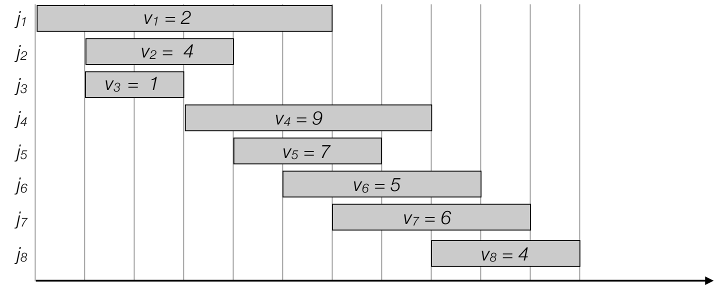
time $\Theta(n)$

space $\Theta(1)$

Weighted Interval Scheduling

Weighted interval scheduling

- Weighted interval scheduling problem
  - n jobs (intervals)
  - Job *i* starts at s_i, finishes at f_i and has weight/value v_i.
  - Goal: Find maximum weight subset of non-overlapping (compatible) jobs.



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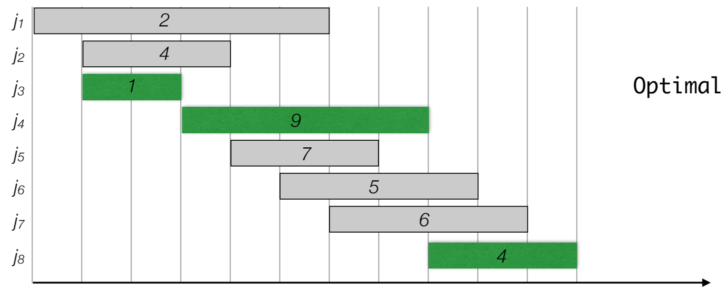
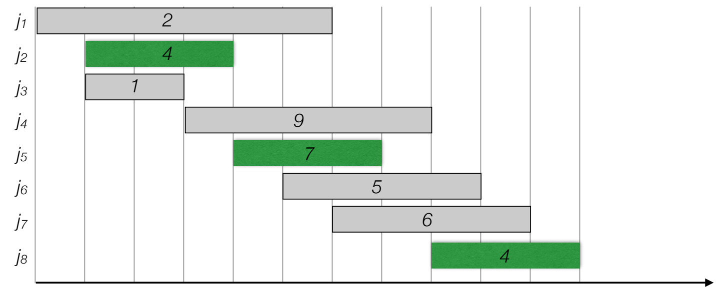


Optimal?

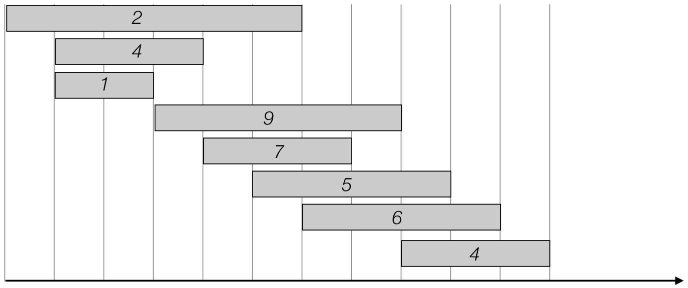
Weighted interval scheduling

- Weighted interval scheduling problem
  - n jobs (intervals)
  - Job *i* starts at s_i, finishes at f_i and has weight/value v_i.
  - Goal: Find maximum weight subset of non-overlapping (compatible) jobs.



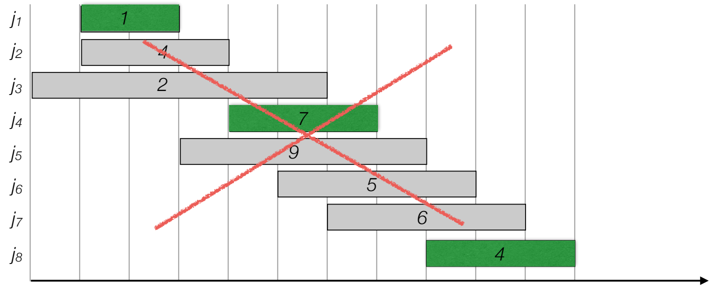
Weighted interval scheduling

- Label/sort jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$



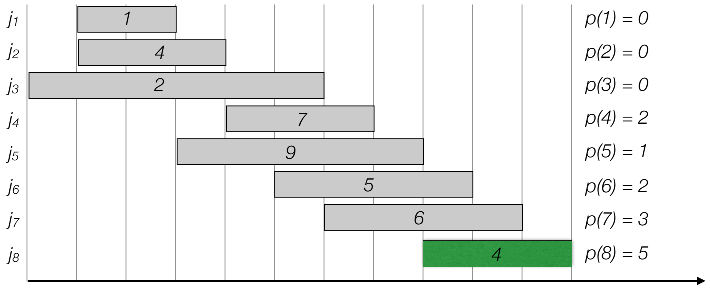
Weighted interval scheduling

- Label/sort jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$
- ~~Greedy?~~



Weighted interval scheduling

- Label/sort jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$
- $p(j)$ = largest index $i < j$ such that job i is compatible with j.
- Optimal solution OPT:
  - Case 1. OPT selects last job
    OPT = v_n + optimal solution to subproblem on 1,…,p(n)
  - Case 2. OPT does not select last job
    OPT = optimal solution to subproblem on 1,…,n-1



Weighted interval scheduling

- OPT(j) = value of optimal solution to the problem consisting job requests 1,2,..,*j*.
  - Case 1. OPT(*j*) selects job j
    OPT(j) = v_j + optimal solution to subproblem on 1,…,p(j)
  - Case 2. OPT(*j*) does not select job j
    OPT = optimal solution to subproblem 1,…j-1
- Recursion:

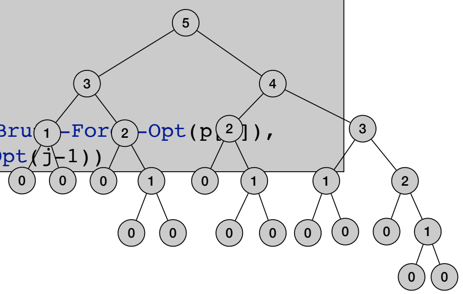
$$OPT(j) = \begin{cases} 0 & \texttt{if } j = 0 \\ \max\{v_j + OPT(p(j)), OPT(j-1)\} & \texttt{otherwise} \end{cases}$$

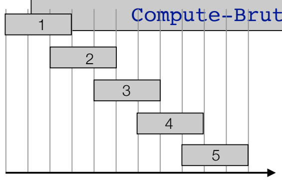
Weighted interval scheduling: brute force

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\{v_j + OPT(p(j)), OPT(j-1)\} & \text{otherwise} \end{cases}$$

```
Input: n, s[1..n], f[1..n], v[1..n]

Sort jobs by finish time so that f[1] ≤ f[2]≤ … ≤ f[n]
Compute p[1], p[2], …, p[n]
Compute-BruteForce-Opt(n)

Compute-Brute-Force-Opt(j)
if j = 0
  return 0
else
  return max(v[j] + Compute-Brute-Force-Opt(p[j]),
             Compute-Brute-Force-Opt(j-1))
```

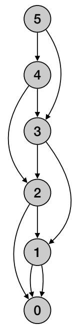
time $\Theta(2^n)$





Weighted interval scheduling: memoization

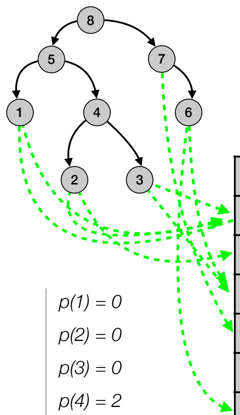
```
Input: n, s[1..n], f[1..n], v[1..n]

Sort jobs by finish time so that f[1] ≤ f[2]≤ … ≤ f[n]
Compute p[1], p[2], …, p[n]

for j=1 to n
  M[j] = null
M[0] = 0.
Compute-Memoized-Opt(n)

Compute-Memoized-Opt(j)
if M[j] is empty
  M[j] = max(v[j] + Compute-Memoized-Opt(p[j]),
             Compute-Memoized-Opt(j-1))
return M[j]
```



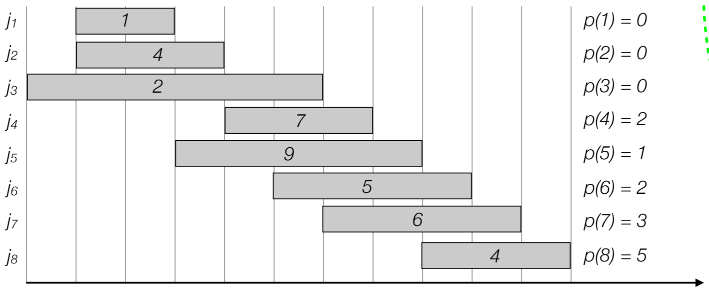
- Running time O(n log n):
  - Sorting takes O(n log n) time.
  - Computing p(n): O(n log n) - use log n time to find each p(i).
  - Each subproblem solved once.
  - Time to solve a subproblem constant.
- Space O(n)

Weighted interval scheduling: memoization

```
Input: n, s[1..n], f[1..n], v[1..n]

Sort jobs by finish time so that f[1] ≤ f[2]≤ … ≤ f[n]
Compute p[1], p[2], …, p[n]

for j=1 to n
  M[j] = empty
M[0] = 0.
Compute-Memoized-Opt(n)

Compute-Memoized-Opt(j)
if M[j] is empty
  M[j] = max(v[j] + Compute-Memoized-Opt(p[j]),
             Compute-Memoized-Opt(j-1))
return M[j]
```



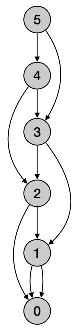
i	M[i]
0	0
1	1
2	4
3	4
4	11
5	11
6	11
7	11
8	15



$p(1) = 0$
$p(2) = 0$
$p(3) = 0$
$p(4) = 2$
$p(5) = 1$
$p(6) = 2$
$p(7) = 3$
$p(8) = 5$

Weighted interval scheduling: bottom-up

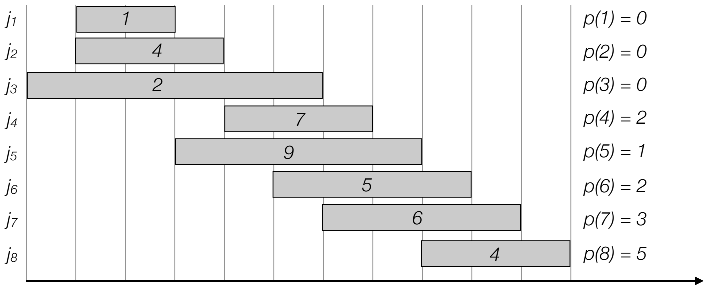
```
Compute-Bottom-Up-Opt(n, s[1..n], f[1..n], v[1..n])

Sort jobs by finish time so that f[1] ≤ f[2]≤ … ≤ f[n]
Compute p[1], p[2], …, p[n]

M[0] = 0.
for j=1 to n
  M[j] = max(v[j] + M(p[j]), M(j-1))
return M[n]
```



- Running time O(n log n):
  - Sorting takes O(n log n) time.
  - Computing p(n): O(n log n)
  - For loop: O(n) time
    - Each iteration takes constant time.
- Space O(n)

Weighted interval scheduling: bottom-up

```
Compute-Bottom-Up-Opt(n, s[1..n], f[1..n], v[1..n])

Sort jobs by finish time so that f[1] ≤ f[2]≤ … ≤ f[n]
Compute p[1], p[2], …, p[n]

M[0] = 0.
for j=1 to n
  M[j] = max(v[j] + M(p[j]), M(j-1))
return M[n]
```

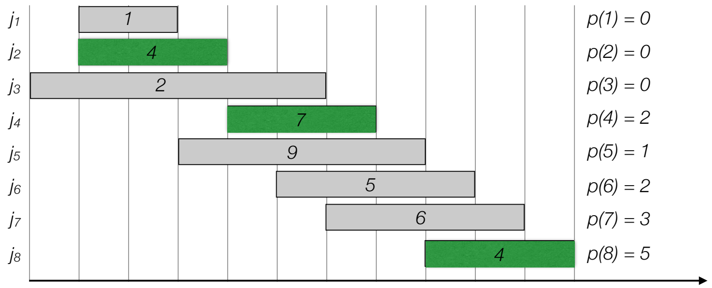


i	M[i]
0	0
1	1
2	4
3	4
4	11
5	11
6	11
7	11
8	15

Weighted interval scheduling: find solution

```
Find-Solution(j)
if j=0
  Return emptyset
else if M[j] > M[j-1]
  return {j} ∪ Find-Solution(p[j])
else
  return Find-Solution(j-1)
```

Solution = 8 , 4 , 2



i	M[i]
0	0
1	1
2	4
3	4
4	11
5	11
6	11
7	11
8	15

Subset Sum and Knapsack

Subset Sum

- Subset Sum
  - Given *n* items $\{1, \ldots, n\}$
  - Item i has weight w_i
  - Bound W
  - Goal: Select maximum weight subset S of items so that

$$\sum_{i \in S} w_i \leq W$$

- Example
  - {2, 5, 8, 9, 12, 18} and W = 25.
  - Solution: 5 + 8 + 12 = 25.

Subset Sum

- $\mathcal{O}$ = optimal solution
- Consider element n.
  - Either in $\mathcal{O}$ or not.
    - $n \notin \mathcal{O}$: Optimal solution using items $\{1,\ldots,n-1\}$ is equal to $\mathcal{O}$.
    - $n \in \mathcal{O}$: Value of $\mathcal{O} = w_n$ + weight of optimal solution on $\{1,\ldots,n-1\}$ with capacity $W - w_n$.
- Recurrence
  - $\text{OPT}(i, w)$ = optimal solution on $\{1,\ldots,i\}$ with capacity w.
  - From above:

$$\text{OPT}(n, W) = \max(\text{OPT}(n-1, W), w_n + \text{OPT}(n-1, W - w_n))$$

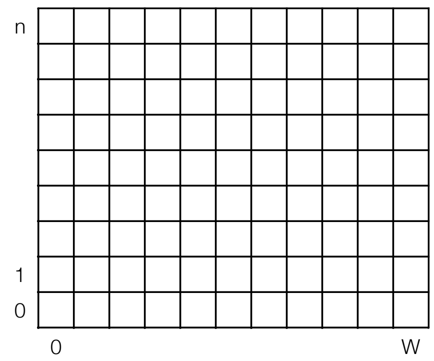
  - If $w_n > W$:

$$\text{OPT}(n, W) = \text{OPT}(n-1, W)$$

Subset Sum

- Recurrence:

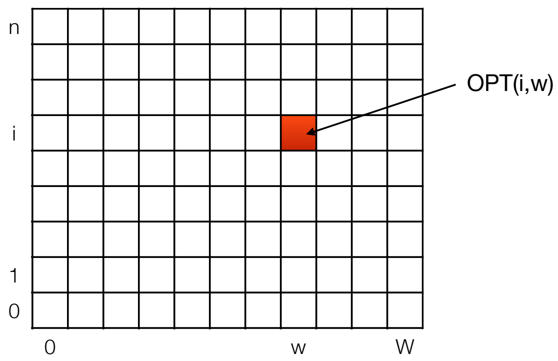
$$\text{OPT}(i, w) = \begin{cases} \text{OPT}(i-1, w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w - w_i)) & \text{otherwise} \end{cases}$$



Subset Sum

- Recurrence:

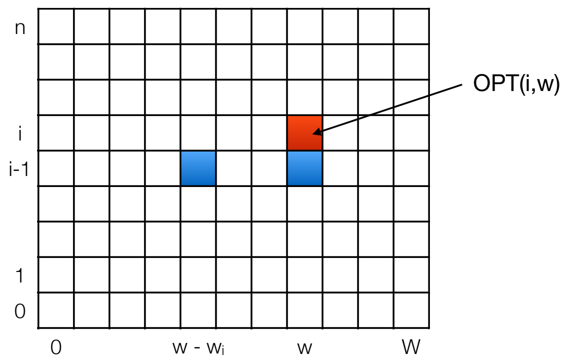
$$\text{OPT}(i, w) = \begin{cases} \text{OPT}(i-1, w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w - w_i)) & \text{otherwise} \end{cases}$$



Subset Sum

- Recurrence:

$$\text{OPT}(i, w) = \begin{cases} \text{OPT}(i-1, w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w - w_i)) & \text{otherwise} \end{cases}$$

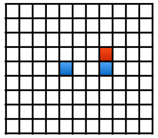


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w),\, w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

```
Subset-Sum(n,W)
  Array M[0…n, 0…W]
  Initialize M[0,w] = 0 for each w = 0,1,…,W
  for i = 1 to n
    for w = 0 to W
      if w < wᵢ
        M[i,w] = M[i-1,w]
      else
        M[i,w] = max(M[i-1,w], wᵢ + M[i-1, w-wᵢ])
  return M[n,W]
```



Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w),\, w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

9	5													
8	4													
5	3													
2	2													
1	1													
-	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	1	2	3	4	5	6	7	8	9	10	11	12

Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w),\, w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

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9	5													
8	4													
5	3													
2	2													
1	1	0												
-	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	1	2	3	4	5	6	7	8	9	10	11	12

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- Example
  - {1, 2, 5, 8, 9} and W = 12

9	5													
8	4													
5	3													
2	2													
1	1	0												
-	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	1	2	3	4	5	6	7	8	9	10	11	12

Subset Sum

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$$\text{OPT}(i, w) = \begin{cases} \text{OPT}(i-1, w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w - w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

9	5													
8	4													
5	3													
2	2													
1	1	0	1											
-	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	1	2	3	4	5	6	7	8	9	10	11	12

Subset Sum

- Recurrence:

$$\text{OPT}(i, w) = \begin{cases} \text{OPT}(i-1, w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w - w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

9	5													
8	4													
5	3													
2	2													
1	1	0	1	1										
-	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	1	2	3	4	5	6	7	8	9	10	11	12

Subset Sum

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- Example
  - {1, 2, 5, 8, 9} and W = 12

9	5													
8	4													
5	3													
2	2													
1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
-	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	1	2	3	4	5	6	7	8	9	10	11	12

Subset Sum

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$$\text{OPT}(i, w) = \begin{cases} \text{OPT}(i-1, w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w - w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

9	5													
8	4													
5	3													
2	2	0												
1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
-	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	1	2	3	4	5	6	7	8	9	10	11	12

Subset Sum

- Recurrence:

$$\text{OPT}(i, w) = \begin{cases} \text{OPT}(i-1, w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w - w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

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8	4													
5	3													
2	2	0	1											
1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
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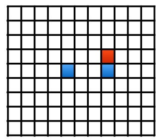
- Running time:
  - Number of subproblems = nW
  - Constant time on each entry $\Rightarrow O(nW)$
  - *Pseudo-polynomial time.*
    - Not polynomial in input size:
      - whole input can be described in O(n log n + n log w) bits, where w is the maximum weight (including W) in the instance.



Knapsack

- Knapsack
  - Given n items $\{1,\ldots,n\}$
  - Item i has weight w_i and value v_i
  - Bound W
  - Goal: Select maximum *value* subset S of items so that
    $$\sum_{i \in S} w_i \leq W$$
- Example

Optimal solution: {3,4} has value 40

value	1	6	18	22	28
weight	1	2	5	6	7

Capacity 11

Knapsack

- $\mathcal{O}$ = optimal solution
- Consider element n.
  - Either in $\mathcal{O}$ or not.
    - $n \notin \mathcal{O}$: Optimal solution using items $\{1,\ldots,n-1\}$ is equal to $\mathcal{O}$.
    - $n \in \mathcal{O}$: Value of $\mathcal{O}$ = v_n + value on optimal solution on $\{1,\ldots,n-1\}$ with capacity $W - w_n$.
- Recurrence
  - OPT(i, w) = optimal solution on $\{1,\ldots,i\}$ with capacity w.

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), v_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Running time $O(nW)$

Dynamic programming

- **First formulate the problem recursively.**
  - Describe the *problem* recursively in a clear and precise way.
  - Give a recursive formula for the problem.
- **Bottom-up**
  - Identify all the subproblems.
  - Choose a memoization data structure.
  - Identify dependencies.
  - Find a good evaluation order.
- **Top-down**
  - Identify all the subproblems.
  - Choose a memoization data structure.
  - Identify base cases.