Synthetic Completeness Proofs for Seligman-style Tableau Systems

Klaus Frovin Jørgensen

Section of Philosophy and Science Studies, Roskilde University

Patrick Blackburn

Section of Philosophy and Science Studies, Roskilde University

Thomas Bolander

Department of Applied Mathematics and Computer Science, Technical University of Denmark

Torben Braüner

Department of People and Technology, Roskilde University

Abstract

Hybrid logic is a form of modal logic which allows reference to worlds. We can think of it as 'modal logic with labelling built into the object language' and various forms of labelled deduction have played a central role in its proof theory. Jerry Seligman's work [13,14] in which 'rules involving labels' are rejected in favour of 'rules for all' is an interesting exception to this. Seligman's approach was originally for natural deduction; the authors of the present paper recently extended it to tableau inference [1,2]. Our earlier work was syntactic: we showed completeness by translating between Seligman-style and labelled tableaus, but our results only covered the minimal hybrid logic; in the present paper we provide completeness results for a wider range of hybrid logics and languages. We do so by adapting the synthetic approach to tableau completeness (due to Smullyan, and widely applied in modal logic by Fitting) so that we can directly build maximal consistent sets of tableau blocks.

Keywords: Hybrid logic, tableaus, Seligman-style, synthetic completeness method, Bridge rule, pure axioms, tense logic, universal modality, difference operator

1 Introduction

Hybrid logic is a form of modal logic which allows us to refer to worlds: it contains nominals (special atomic formulas true at a unique world) and formulas of the form $@_i\varphi$. Here i is a nominal, and the formula $@_i\varphi$ is true iff φ is true at the unique world where i is true. So hybrid logic can be thought of as modal logic with world-labelling apparatus hard-wired into the object language, and various forms of labelled deduction have played the leading role in the development of hybrid proof theory. That is, many hybrid proof systems work by manipulating formulas of the form $@_i\varphi$ (and manipulating *only* formulas of this form) rather than arbitrary formulas. [1]

An exception to this is Jerry Seligman's work (dating back to the 1990s) in which 'rules involving labels' are rejected in favour of 'rules for all'. Seligman introduced his approach in two papers, the natural deduction based [13] and the sequent calculus based [14]; his natural deduction approach was later developed by Braüner [4]. We recently adapted Seligman's approach to tableau inference [1,2]; our key idea was to subdivide tableau branches into *blocks*, and use a rule called GoTo to navigate between them. Our investigations were syntactic: we proved completeness by explicit translation between Seligman-style and labelled tableaus, but only discussed the minimal hybrid logic.

In this paper we provide completeness results for a wider range of hybrid logics and languages. In Section 2 we introduce STB and ST, the two Seligman-style tableau systems we shall work with. [2] In Section 3 we adapt the synthetic tableau completeness method so that we can directly build maximal consistent sets of blocks, rather than maximal consistent sets of formulas or labelled formulas; [3] this yields completeness for STB. In Section 4 we eliminate a rule called Bridge to obtain completeness for ST. In Section 5 we show that this leads to completeness for richer logics and languages. Section 6 concludes.

2 Two Basic Calculi: ST and STB

We mostly work with a basic hybrid language built over a countable set of propositional symbols and a countable set of nominals. We take $\neg$, $\vee$, $\Diamond$, and for each nominal i an $@_i$-operator as primitive connectives, and build formulas as follows (here i ranges over nominals and p over propositional symbols):

$$\varphi ::= i \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi \mid @_i\varphi.$$

Other booleans are defined as usual, and $\Box\varphi$ is defined to be $\neg\Diamond\neg\varphi$. Note that nominals can occur either as subscripts to @ ("in operator position") or as formulas in their own right ("in formula position"). We typically use i, j and k for nominals and p, q and r for ordinary propositional symbols. Nominals and propositional symbols are the atomic formulas.

[1] Labelled deduction has a long history in modal logic, not just hybrid logic. The basic idea dates back to Fitch [6], was developed and generalized by Fitting [7,8,9], used for intuitionistic modal logic by Simpson [15], and became a proof strategy for many kinds of non-classical logics with the work of Gabbay [10]. Work by Negri has taken the approach in another direction by grounding it in the Kleene-Gentzen G3 system; see, for example, [12].

[2] Space limitations mean we cannot further discuss our reasons for finding Seligman-style tableaus so interesting, though we give motivating examples in figures 3 and 8. For a deeper discussion of Seligman's work and its links with our own, we refer the reader to [2].

[3] The synthetic method dates back to Smullyan's [16] classic work on first-order tableaus, and has been widely employed in modal logic by Fitting.

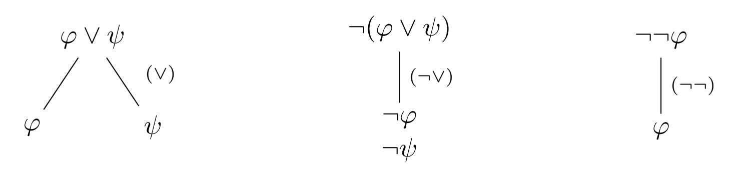


Fig. 1. Tableau rules for propositional logic.

We interpret the language in models based on frames (W, R), where W is a non-empty set (we call its elements worlds) and R is a binary relation on W (the accessibility relation). A model is a triple (W, R, V) where (W, R) is a frame and V (the valuation) maps propositional symbols p to arbitrary subsets of W, and nominals i to singleton subsets of W. A model is *named* iff for every world w there is some nominal i such that $V(i) = \{w\}$.[4]

Satisfiability in a model is defined in the usual way as a relation between a model $\mathfrak{M} = (W, R, V)$, a world $w \in W$, and a formula φ:

$$\begin{array}{ll}
\mathfrak{M}, w \models a & \text{iff } a \text{ is atomic and } w \in V(a) \\
\mathfrak{M}, w \models \neg\varphi & \text{iff } \mathfrak{M}, w \not\models \varphi \\
\mathfrak{M}, w \models \varphi \vee \psi & \text{iff } \mathfrak{M}, w \models \varphi \text{ or } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \Diamond\varphi & \text{iff for some } w',\ wRw' \text{ and } \mathfrak{M}, w' \models \varphi \\
\mathfrak{M}, w \models @_i\varphi & \text{iff } \mathfrak{M}, w' \models \varphi \text{ and } w' \in V(i).
\end{array}$$

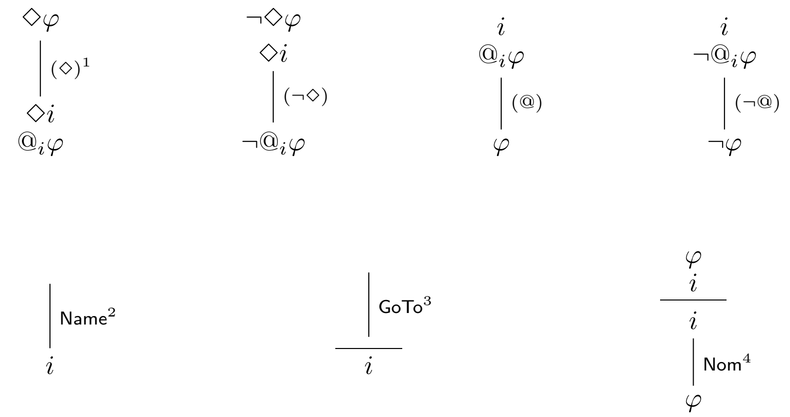
A formula φ is *valid on* $\mathfrak{M} = (W, R, V)$ when for all worlds $w \in W$ we have that $\mathfrak{M}, w \models \varphi$. A formula is *valid* if it is valid on all models.

Now for our tableau systems. For the propositional connectives we use the standard rules shown in figure 1. And we work with the usual notion of branch in a (tableau) tree. But we also need *blocks*: given a branch Θ in a tableau, we define a block to be one of the following:

- The *initial block*, consisting of all the formulas on Θ until the first horizontal line (or all formulas if there is no such line).
- The *current block*, consisting of all formulas below the last horizontal line (or all formulas if there is no such line).
- All formulas that occur between a pair of two consecutive horizontal lines.

The rule allowing us to close down one block and start up a new one is GoTo. Its precise formulation is given in figure 2. All blocks except the initial one are opened by an application of GoTo, and hence they all contain a nominal as

[4] That is, a model is named if each of its worlds is named by some nominal. Most models are not named (as our language is countable) but as we shall see in Section 5, named models have desirable properties. Our goal here is to prove completeness by building named models. Note: distinct nominals can name the same world (just as distinct first-order constants can denote the same element of a first-order model).



[1] The nominal i is fresh and φ is not a nominal.
[2] The nominal i is fresh.
[3] The nominal i must already occur on the branch.
[4] The horizontal line below the two uppermost premises signifies that these premises belong to a block distinct from the current one, whereas the third premise (the lowermost occurrence of i) belongs to the current block.

Fig. 2. ST tableau rules for basic hybrid logic.

their first formula. This nominal is called the *opening nominal* of the block. If the initial block contains one or more nominals generated by applications of the Name rule (see figure 2), then the nominal generated by the first application will be called the *opening nominal* of the initial block. Otherwise, the initial block will have no opening nominal.

The crucial rules of the Seligman-style tableau calculi are given in figure 2. The general conditions on rule applications are as follows:

- The propositional rules $(\vee)$, $(\neg\vee)$, $(\neg\neg)$ as well as $(\Diamond)$ and $(\neg\Diamond)$ can only be applied to premises that belong to the current block or a previous block with the same opening nominal.
- In the rules $(@)$ and $(\neg @)$, the first premise i must either belong to the current block or a previous block with the same opening nominal. The second premise $@_i\varphi$ $(\neg @_i\varphi)$ can appear anywhere on the branch.
- GoTo and Name can always be applied as they have no premises.
- Nom can be applied as described in the rule itself: if φ and i belong to some block distinct from the current block, and i belongs to the current block, then φ can be added to the current block.

In figure 3 we illustrate the calculus with two simple tableaus: one proves a valid formula, the other provides a counter-example for a non-valid formula.

1	$\neg(@_i j \wedge @_j \varphi \to @_i \varphi)$	
2	$@_i j \wedge @_j \varphi$	$(\neg \to)$ on 1
3	$\neg @_i \varphi$	$(\neg \to)$ on 1
4	$@_i j$	$(\wedge)$ on 2
5	$@_j \varphi$	$(\wedge)$ on 2
6	i	GoTo
7	j	$(@)$ on 4,6
8	φ	$(@)$ on 5,7
9	$\neg \varphi$	$(\neg @)$ on 3,6
	$\times$	

1	$\neg(\Diamond\Diamond i \to \Diamond i)$	
2	k	Name
3	$\Diamond\Diamond i$	$(\neg \to)$ on 1
4	$\neg \Diamond i$	$(\neg \to)$ on 1
5	$\Diamond j$	$(\Diamond)$ on 3
6	$@_j \Diamond i$	$(\Diamond)$ on 3
7	$\neg @_j i$	$(\neg\Diamond)$ on 4,5
8	j	GoTo
9	$\Diamond i$	$(@)$ on 6,8
10	$\neg i$	$(\neg @)$ on 7,8

Fig. 3. On the left the valid $@_j i \wedge @_j \varphi \to @_i \varphi$ is proved. On the right there is a non-closed tableau which can be used to construct a non-transitive model falsifying $\Diamond\Diamond i \to \Diamond i$. The model has three worlds, $\overline{k}$, $\overline{j}$ and $\overline{i}$, where only $(\overline{k}, \overline{j})$ and $(\overline{j}, \overline{i})$ are in the accessibility relation. The Name-rule gives us a name for the first world. In both tableaus we use derived rules for defined connectives.

Our first Seligman-style tableau calculus is called ST and consists of the rules given in figure 1 and 2. Tableaus are built in the expected way, but let us be explicit about our closure condition: *a branch closes either by having φ and $\neg\varphi$ inside a block, or inside two distinct blocks with the same opening nominal.* In [2] we showed that ST could prove all validities by translating between ST-tableaus and labelled tableaus. Here our goal is to prove a completeness result for ST that can be straightforwardly generalised to richer logics and languages.

To do this we will introduce a second calculus called STB. This is ST augmented by the Bridge rule, which is shown in figure 4. For the rule to be applicable three conditions have to be satisfied: (1) i and $\Diamond j$ must occur together on the same block, or on blocks with the same opening nominal; (2) j and k also have to occur on the same block, or on blocks with the same opening nominal; and (3) i must occur on the current block or on a block with the same opening nominal as the current block. Note that Bridge can be seen a restricted form of cut for nominals: reading it from bottom to top, it lets us 'cut' by introducing new symbols j and $\Diamond j$. We first prove completeness for STB, and then extend it to a proof for ST by eliminating Bridge.

3 Completeness for STB

In this section we adapt the Smullyan-Fitting synthetic completeness method so that it works with maximal consistent sets of *blocks* (rather than merely formulas or labelled formulas). This leads directly to completeness for STB.

3.1 Hintikka sets of blocks and induced models

First we generalize the notion of Hintikka set. Our tableau branches are sequences of finite blocks of formulas, each of which (with the possible exception of the first) is named by an opening nominal. So the first step is to choose a set-theoretic representation of this fundamental notion. Accordingly we define a *named block* to be a pair (A, i), where A is a set of formulas that has the

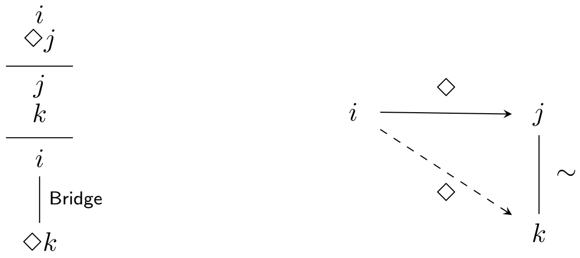


Fig. 4. The Bridge rule is shown on the left. On the right we show its semantic counterpart: if there is an edge from i to j, and if j and k are equivalent (that is, they name the same world) then there is an edge from i to k. This notion of equivalence is made precise as $\sim_H$ in the forthcoming model-constructions.

nominal i as one of its elements. So named blocks are pointed sets, and when we talk of an *i-block* we mean a named block of the form (A, i), that is, a named block with i as its distinguished element. We say that a formula φ is *on* a named block (A, i) iff $\varphi \in A$. A named block (A, i) is finite iff A is finite.

A set of finite named blocks H is called a *Hintikka set* if it satisfies the following properties:

(i) If there is an i-block in H with an atomic formula a on it, then there is no i-block in H with $\neg a$ on it.

(ii) If there is an i-block in H with $\Diamond j$ on it, then there is no i-block in H with $\neg\Diamond j$ on it.

(iii) If φ is a formula containing an occurrence of nominal i (either in formula or operator position) and φ is on some named block in H, then there is an i-block in H.

(iv) If there is an i-block in H with j on it, there is a j-block with i on it too.

(v) If there is an i-block in H with j on it, and a j-block in H with k on it, then there is an i-block in H with k on it.

(vi) If there is an i-block in H with $\Diamond j$ on it, and a j-block in H with k on it, then there is an i-block in H with $\Diamond k$ on it.

(vii) If there is an i-block in H with $\Diamond j$ on it, and an i-block in H with k on it, then there is an k-block in H with $\Diamond j$ on it.

(viii) If there is an i-block in H with $\varphi \vee \psi$ on it, then there is an i-block in H with φ or ψ on it.

(ix) If there is an i-block in H with $\neg(\varphi \vee \psi)$ on it, then there is an i-block in H with both $\neg\varphi$ and $\neg\psi$ on it.

(x) If there is an i-block in H with $\neg\neg\varphi$ on it, then there is an i-block in H with φ on it.

(xi) If there is a named block in H with $@_j\varphi$ (or $\neg @_j\varphi$) on it, then there is a j-block in H with φ (or $\neg\varphi$) on it.

(xii) If there is an i-block in H with $\Diamond\varphi$ on it, and φ is not a nominal, then there are (possibly identical) i-blocks in H with $\Diamond j$ and $@_j\varphi$ on them.

(xiii) If there are (possibly identical) i-blocks in H with $\neg\Diamond\varphi$ and $\Diamond j$ on them, then there is an i-block in H with $\neg @_j\varphi$ on it.

Think of a Hintikka set H as an (abstract versions of) an exhausted open tableau branch. We will construct a model out of equivalence classes of the nominals that name its blocks. If nominals i and j are names of blocks on H, we define $i \sim_H j$ iff there is an i-block in H with j on it; it follows from the iiird, ivth and vth Hintikka properties that $i \sim_H j$ is an equivalence relation. For a Hintikka set H of finite named blocks and a nominal i, we denote the equivalence class of i by $|i|_{\sim_H}$, suppressing $\sim_H$ when it's clear from context.

Given a Hintikka set H we construct *the named model* $\mathfrak{M}_H$ *induced by* H as follows. The set of worlds W_H of $\mathfrak{M}_H$ is the set of equivalence classes of nominals occurring in H. The relation R_H of $\mathfrak{M}_H$ is defined by:

$$|i|R_H|j| \text{ iff there is an } i\text{-block in } H \text{ with } \Diamond j \text{ on it.}$$

The well-definedness of R_H follows from the combination of the vith and viith Hintikka properties. The valuation function V_H of $\mathfrak{M}_H$ is defined by:

(i) If nominal i is in an equivalence class in W_H, set $V_H(i) = \{|i|\}$. Otherwise pick some $|j| \in W_H$ and set $V_H(i) = \{|j|\}$. Clearly V_H names every world.

(ii) If propositional variable p occurs on some block in H, then set $V_H(p)$ to be $\{|i| \ : \ \text{there is an } i\text{-block in } H \text{ with } p \text{ on it}\}$. Otherwise set $V_H(p) = \emptyset$.

Lemma 3.1 (Hintikka block lemma) *Any Hintikka set H is satisfied by the named model it induces.*

Proof. Suppose H is a Hintikka set of finite named blocks. Let $\mathfrak{M} = (W, R, V)$ be the named model induced by H. Then by simultaneous induction on the complexity of φ we can show the following (stronger) fact:

A. If φ is on some i-block in H, then $\mathfrak{M}, |i| \models \varphi$,

B. If $\neg\varphi$ is is on some i-block in H, then $\mathfrak{M}, |i| \not\models \varphi$.

We need the stronger formulation to drive the inductive step for $\neg$ through. □

3.2 Maximal consistent sets of blocks

Let S be a finite set of finite named blocks. We say that T is a *named* `STB`-*tableau for S* when the initial part of T consists of the finite named blocks from S and the rest of T is constructed by application of the `STB`-rules. If T is a named `STB`-tableau for a set S of finite named blocks, any block in S is called a *root-block*; note that the order of the root blocks in the initial part of T doesn't matter. We then say that a (possibly infinite) set S of finite named blocks is `STB`-*consistent* when there doesn't exist a closed `STB`-tableau for any finite subset of S. We say that a set $S^\star$ of finite named blocks is *maximally* `STB`-*consistent* if is `STB`-consistent, and no proper extension of $S^\star$ is `STB`-consistent.

To prove completeness for `STB` (and later `ST`) we will need two technical lemmas. Let $B[i/j]$ denote the uniform substitution of nominal i for j in block

B; this means that we have uniformly substituted i for j in every formula occurring in B. We can also substitute i for j in a branch Θ if we substitute i for j in every block B occurring in Θ; this is written $\Theta[i/j]$.

Lemma 3.2 (Substitution lemma)

(i) ***We can uniformly substitute in a tableau.*** *Suppose Φ is a branch in an* `STB`*-tableau in which the nominals i and j occur, and Φ is extended by applying rule R to input I to obtain output O. Then $\Phi[i/j]$ can be extended by applying R to input $I[i/j]$ to obtain output $O[i/j]$.*

(ii) ***Which nominals are used as fresh nominals is irrelevant.*** *Suppose Θ is a branch in a finite tableau T and let i be a nominal which is used somewhere in Θ as a fresh nominal. Suppose, furthermore, that j is not used at all in Θ. We can then uniformly substitute j for i in Θ to obtain a branch $\Theta[j/i]$, where rule-applications in $\Theta[j/i]$ mimic rule-applications in Θ, the only change being that j is used instead of i.*

Proof. By induction on the construction of Φ and Θ respectively. □

We use the following lemma repeatedly (without comment) in what follows.

Lemma 3.3

(i) *If there exists a closed tableau for a finite set S^f of finite named blocks, then there exists a closed tableau for any finite set of finite named blocks which is a superset of S^f.*

(ii) *If S is a maximal* `STB`*-consistent set of finite named blocks, and if there is an i-block in S with φ on it, then there is also an i-block in S with just φ and i on it.*

Proof. Straightforward with the help of the substitution lemma. □

We are now ready for a version of the *Lindenbaum-Henkin construction* that works for named blocks. First, assume we have an enumerated infinite set of fresh nominals; these will be used to witness the diamonds. [5] Second, we will need an enumeration of the finite named blocks of this extended language; let's assume that these are given as $B_1, B_2, B_3, \ldots$, and so on.

Let S be an `STB`-consistent set of finite named blocks in the original (non-extended) language. We now construct, starting with S, an increasing sequence of consistent sets of finite named blocks such that the union of the whole sequence is our desired maximal `STB`-consistent set (in the extended language). But we need to ensure, that the diamonds are witnessed in the final union. So if some $\Diamond\varphi$ is on some block B and φ is not a nominal, we will create a block B' called the $\Diamond$-witness for B, such that for any $\Diamond\varphi$ on B, with φ not a nominal, $\Diamond i$ and $@_i\varphi$ are on B', with i being the first fresh nominal. The

[5] We say "witness" as we can view $\Diamond$ as an existential quantifier over acccessible worlds. We will witness diamond formulas with nominals—thereby mimicking Henkin's well-known first-order completeness proof strategy of witnessing existential formulas with constants.

Lindenbaum-Henkin-construction then goes like this: Let S_1 be S. Suppose S_n has been constructed. Then:

$$S_{n+1} = \begin{cases} S_n, & \text{if } S_n \cup \{B_n\} \text{ is inconsistent,} \\ S_n \cup \{B_n\}, & \text{if } S_n \cup \{B_n\} \text{ is consistent, and on } B_n \text{ there is} \\ & \text{no } \Diamond\varphi, \\ S_n \cup \{B_n\} \cup \{B'\}, & \text{if } S_n \cup \{B_n\} \text{ is consistent, and on } B_n \text{ there is} \\ & \text{at least one } \Diamond\varphi, \text{ with } \varphi \text{ not a nominal and } B' \\ & \text{ is the } \Diamond\text{-witness for } B_n. \end{cases}$$

Finally, we'll say that a set S of finite named blocks is *$\Diamond$-saturated*, if for any $\Diamond\varphi$ occurring on any i-block $B \in S$, there are (possibly identical) i-blocks B_1 and B_2 with $\Diamond j$ and $@_j\varphi$ on them.

Lemma 3.4 (Lindenbaum-Henkin) *Any* STB*-consistent set of finite named blocks can be extended into a $\Diamond$-saturated maximally* STB*-consistent set of finite named blocks.*

Proof. Let S be a consistent set of finite named blocks. Use S as input for the Lindenbaum-Henkin-construction. The claim we will prove is that $\bigcup S_n$ is a $\Diamond$-saturated maximally STB-consistent set.

We first need to prove that the sequence is a sequence of consistent sets ordered by $\subseteq$. By construction it's clear, that the sequence is ordered by $\subseteq$. It is not difficult, using lemma 3.2.ii, to show that if S_n is consistent then S_{n+1} is consistent too.

Consistency. Suppose $\bigcup S_n$ is not consistent. Then there is a finite set S^f of finite named blocks such that there is a closed STB-tableau for S^f. All the blocks in S^f is found in our enumeration of the finite named blocks. Let m be the largest index number. But then $B \in S_{m+1}$, for every $B \in S^f$, so by lemma 3.3.i we can construct a closed tableau for S_{m+1} which contradicts the consistency of S_{m+1}.

Maximality. Suppose $\bigcup S_n$ isn't maximal. Thus for some m, $B_m \notin \bigcup S_n$ and $\bigcup S_n \cup B_m$ is consistent. But $S_{m+1} \subseteq \bigcup S_n$ and $S_{m+1} = S_m \cup \{B_m\}$, (as $S_m \cup B_m$ is also consistent). But $B_m \in \bigcup S_n$, as $B_m \in S_{m+1} \subseteq \bigcup S_n$.

$\Diamond$-saturatedness. Suppose $\Diamond\varphi$ occurs on some i-block $B \in \bigcup S_n$. This block B occurs somewhere in the enumeration; suppose it's indexed by m. Then by the construction of S_{m+1} we have a witnessing i-block B' with both $\Diamond j$ and $@_j\varphi$ on B'. As $S_{m+1} \subseteq \bigcup S_n$ we have $B' \in \bigcup S_n$. □

3.3 Completeness

We have reached the heart of the synthetic completeness method:

Lemma 3.5 (Smullyan-Fitting block lemma for STB) *If S is a maximal* STB*-consistent set of finite named blocks, then S is a Hintikka set.*

Proof. Given a maximal STB-consistent set S of finite named blocks, we can show that all Hintikka properties hold for S. Here we prove four cases, including property vi, which is where Bridge is used.

Property i and ii. Let φ be either a nominal j, a propositional symbol p or the formula $\Diamond j$. Assume there is an i-block $B_1 \in S$ with φ on it. Further suppose that there is another i-block $B_2 \in S$ with $\neg\varphi$ on it. Then from the union of B_1 and B_2 we have a closed tableau, as both φ and $\neg\varphi$ occur on two (possibly identical) i-blocks, contradicting the STB-consistency of S.

Property iii. Assume that i occurs in some formula on some block $B_1 \in S$, but assume—working towards a contradiction—that there is no i-block at all in S. Let B_2 be the i-block which only has i on it. As B_2 is not in S it follows by the maximality of S that $S \cup \{B_2\}$ is STB-inconsistent. So there is a finite set S^f which is a subset of S such that there exists a closed tableau for $S^f \cup \{B_2\}$. Now, take S^f and add B_1 to it (if it isn't already there). Using GoTo we can extend $S^f \cup \{B_1\}$ by opening an i-block, and by copying the tableau we have for $S^f \cup \{B_1\}$ we can extend this to a closed tableau. But this contradicts the STB-consistency of S.

Property vi. Suppose there is an i-block $B_1 \in S$ with $\Diamond j$ on it and a j-block $B_2 \in S$ with k on it. Further suppose that there is no i-block with $\Diamond k$ on it in S. Let B_3 be the i-block consisting just of i and $\Diamond k$. By the maximality of S we have that $S \cup \{B_3\}$ is inconsistent. Thus, for a finite $S^f \subseteq S$ there is a closed tableau T for $S^f \cup \{B_3\}$. Now take $S^f \cup \{B_1\} \cup \{B_2\}$ as root-blocks. We start a tableau for these root-blocks by applying GoTo to open an i-block, and then apply Bridge to add $\Diamond k$ to it. But then we can construct a closed tableau for a finite subset of S which is a contradiction. □

We are ready to prove completeness for STB. We say that a formula φ is STB-consistent iff there is no closed STB tableau with φ as its sole root formula.

Theorem 3.6 *Every* STB*-consistent formula is satisfiable on a named model.*

Proof. Given an STB-consistent formula φ, take it as the root formula of a tableau. Immediately apply rule Name to ensure that our initial block has a name, i say. Applying Name cannot lead to a closed tableau as φ is STB-consistent. Hence $(\{\varphi, i\}, i)$ is an STB-consistent finite named block, and by the Lindenbaum-Henkin lemma it can be extended to a maximal STB-consistent set of finite named blocks S, which by the Smullyan-Fitting block lemma is a Hintikka set. We then form the named model $\mathfrak{M}$ induced by S, and by the Hintikka block lemma have $\mathfrak{M}, |i| \models \varphi$. □

4 Completeness for ST

The only place where Bridge was used in the completeness proof for STB was to show Hintikka property vi in the proof of the Smullyan-Fitting block lemma. We now show that we can prove a version of this lemma without using Bridge, that is, we can prove a Smullyan-Fitting block lemma for ST as well. We do so by showing that whatever we can do in a tableau using an i-block with $\Diamond k$ on it, we can already do using an i-block with $\Diamond j$ on it and a j-block with k on it. This is precisely what is needed for the vith property. To make this work we need to be able to keep track of formula occurrences that stem from or are influenced by $\Diamond k$ on the i-block.[6]

In a named ST-tableau T a formula occurrence of φ *descends from* $\Diamond i$ *in a root-block* B, if the occurrence of φ is:

(i) $\Diamond i$ as it occurs in the root-block B, or

(ii) the output of an application of either $(\neg\Diamond)$ or Nom which as input has a formula occurrence that descends from $\Diamond i$ in the root-block B.

Lemma 4.1 *If an occurrence of φ descends from $\Diamond i$ occurring on a root block B, then φ is either $\Diamond i$ or $\neg @_i \psi$, for some ψ.*

Proof. Let T be a named ST-tableau with root block B having $\Diamond i$ as an element. Suppose φ descends from $\Diamond i$ in B. The lemma is proved by induction on the construction of T.

Base case. Suppose φ is in root block and φ descends from $\Diamond i$ in B. As no rules have been applied, φ can only be $\Diamond i$ in B.

Inductive step. Suppose the lemma holds for T_0 and that T_0 is extended by applying R, thereby extending branch Θ_0 to Θ. If none of the formulas in the output of R descend from $\Diamond i$ on B, the lemma holds trivially for Θ. Suppose therefore that some of rule R's output descends from $\Diamond i$. Which rules could R be? By the definition of descendence, R can only be $(\neg\Diamond)$ or Nom.

Subcase a). Suppose R is $(\neg\Diamond)$ having input that descends from $\Diamond i$. By the induction hypothesis it can only be $\Diamond i$ of the two input-formulas that descends from $\Diamond i$ on B. The output of R, which by assumption descends from $\Diamond i$, is $\neg @_i \psi$, for some ψ. The lemma is thus proved for this case.

Subcase b). Suppose R is Nom. Its input is j and γ, for some nominal j and some formula γ. As the output of R by assumption desends from $\Diamond i$, some of the input of R has to descend (by definition of descendence). So γ, by the induction hypothesis, has to descend. Therefore γ is either $\Diamond i$ or $\neg @_i \psi$, for some ψ. Thus, the output of R which descends from $\Diamond i$ is either $\Diamond i$ or $\neg @_i \psi$, and the lemma is proved for this case. $\square$

We next prove the crucial elimination lemma. First we need a convention. The use in a tableau proof of a $\Diamond i$ occurring in a root block has a possible trace that we need to follow and modify if we want to eliminate this occurrence of $\Diamond i$ from the root block. But recall that only two types of formulas, namely $\Diamond i$ and $\neg @_i \psi$, can descend from $\Diamond i$. This motivates the following:

- $\Diamond j$ is the j-replacement of $\Diamond i$, if the latter descends from $\Diamond i$
- $\neg @_j \psi$ is the j-replacement of $\neg @_i \psi$, if the latter descends from $\Diamond i$

In general, and to establish notation, if φ descends from $\Diamond i$ we denote its j-replacement by φ^j.

[6] Note that our elimination proof does *not* show that Bridge is derivable in ST. Rather, it shows that ST can do something equivalent to Bridge with respect to the blocks B_1, B_2 and B_3 referred to in the proof of Hintikka property vi (page 10). This is why the elimination lemma proved below is formulated in terms of blocks of this form.

Lemma 4.2 (Elimination lemma) *Suppose B_1 is the i-block consisting of i and $\Diamond j$, B_2 is the j-block consisting of j and k, and B_3 is the i-block consisting of i and $\Diamond k$. Suppose furthermore that S is any finite set of finite named blocks. Given a finite* ST*-tableau T for $S \cup \{B_3\}$ we can construct another finite* ST*-tableau T' for $S \cup \{B_1\} \cup \{B_2\}$ such that there is a correspondence between the branches of T and T' in such a way, that given any branch Θ of T, the following holds for any formula φ occurring on any l-block in Θ:*

(i) *If φ does not descend from $\Diamond k$ in B_3, then φ occurs on an l-block of the corresponding Θ' in T'.*

(ii) *If φ descends from $\Diamond k$ in B_3, then φ^j occurs on an l-block of the corresponding Θ' in T'.*

Proof. Let S, B_1, B_2 and B_3 be as supposed in the lemma, and let T be the finite ST-tableau for $S \cup \{B_3\}$. We also suppose that $j \neq k$, as otherwise there is nothing to prove. We can moreover, due to the second part of lemma 3.2, assume that j is not used as a fresh nominal in T. The proof will proceed by induction of the depth of branches of T: We systematically take branches Θ of T and produce corresponding branches Θ' of T', ensuring on the fly that the properties of the lemma hold.

Base case. Out of the root blocks of T we need to construct the root-blocks of T'. This is simple: just replace the T root-block B_3 by B_1 and B_2. Thus, the root blocks of T' are $S \cup \{B_1\} \cup \{B_2\}$. The lemma holds trivially.

Inductive step. Suppose we have gone through the initial part Θ_0 of Θ, the branch we follow in T, and that the lemma holds for Θ_0 and corresponding Θ'_0. Suppose the way Θ_0 is extended is by applying rule R. Either:

a) No formula occurrence of the input for R descends from $\Diamond k$ in B_3, or

b) Some formula occurrence of the input for R descends from $\Diamond k$ in B_3.

Case a). In this case an application of R, which extends Θ_0, will also extend Θ'_0, as all the input for R is to be found, by the induction hypothesis, on Θ'_0.

Case b). Suppose for some φ that is input for rule R which has been used to extend Θ_0, φ descends from $\Diamond k$ in B_3. The induction hypothesis will then give us φ^j on the corresponding Θ'_0. In such cases we cannot generally apply R right away to extend Θ'_0 as the input may not fit anymore; so we'll need to repair such mismatches. Now, which rules could R possibly be? If some input has to descend from $\Diamond k$ on B_3, the input must, by lemma 4.1, be either $\Diamond k$ or $\neg @_k \psi$. By inspection of the ST-rules we then see that R can only be $(\neg\Diamond)$, $(\neg @)$ or Nom. Which subcases will this give rise to? In case the input of $(\neg\Diamond)$ descends from $\Diamond k$, the input has to be $\Diamond k$ and $\neg\Diamond\psi$, for some ψ, both occuring on l-blocks of Θ_0. Analogously, in case input for $(\neg @)$ descends from $\Diamond k$, the input has to be $\neg @_k \psi$ occuring on an m-block and k occuring on an l-block. In case an input formula for Nom descends from $\Diamond k$ it has to be either $\Diamond k$ or $\neg @_k \psi$. Thus there are four subcases here:

Subcase i). The rule-application extending Θ_0 is $(\neg\Diamond)$ which has input $\Diamond k$ occurring on an l-block of Θ_0 and $\neg\Diamond\psi$ occurring on an l-block of Θ_0; the input

$$
\text{IH}\left\{
\begin{array}{rc}
1 & j\\
2 & k\\ \hline
 & \vdots \\ \hline
3 & m\\
4 & \neg @_j\psi\\ \hline
5 & l\\
6 & k\\ \hline
7 & l
\end{array}\right.
$$

$$
\begin{array}{rcl}
8 & k & \mathsf{Nom} \text{ on } 5,6,7\\
9 & j & \mathsf{Nom} \text{ on } 1,2,8\\
10 & \neg\psi & (\neg @) \text{ on } 4,9
\end{array}
$$

Fig. 5. The extension of Θ_0' in subcase ii.

$\Diamond k$ descends from $\Diamond k$ in B_3. The output of R is $\neg @_k\psi$ on the current l-block; this occurrence of $\neg @_k\psi$ also descends from $\Diamond k$ on B_3.

The induction hypothesis gives us the occurrence of the j-replacement $\Diamond j$ on an l-block of the corresponding Θ_0', and $\neg\Diamond\psi$ on an l-block of Θ_0'. We now apply $(\neg\Diamond)$ in order to extend Θ_0' and add $\neg @_j\psi$ on the current l-block. The output of R extending Θ_0 was $\neg @_k\psi$ descending from $\Diamond k$ in B_3. As we have created the j-replacement on the corresponding branch the lemma is proved for this subcase.

Subcase ii). Suppose the rule-application is $(\neg @)$ with the input $\neg @_k\psi$ which descends from $\Diamond k$ in B_3. Suppose furthermore that the input $\neg @_k\psi$ occurs on an m-block and the other input for $(\neg @)$ is k occurring on an l-block. The output of $(\neg @)$ is $\neg\psi$ which is added to the current l-block. Note, the output of R thus extending Θ_0 does not descend from $\Diamond k$ in B_3.

The induction hypothesis gives us a) An m-block on the corresponding Θ_0' with the j-replacement $\neg @_j\psi$, b) an l-block on Θ_0' with k on, and c) the current block is an l-block. figure 5 shows how to extend Θ_0' in this case—note, that we have suppressed quite a few dots.

Subcase iii). Suppose the rule application extending Θ_0 is Nom, where one of the input formulas is $\Diamond k$, which descends from $\Diamond k$ in B_3 and the input $\Diamond k$ occurs toegether with a nominal m on an n-block of Θ_0. The current block is an l-block on which m occurs. Θ_0 is thus extended by adding $\Diamond k$ to the current l-block. This output $\Diamond k$ descends from $\Diamond k$ in B_3.

The induction hypothesis gives us that the current block of the corresponding Θ_0' is an l-block. Moreover, on Θ_0' there is an n-block on which both m and the j-replacement $\Diamond j$ occur. We extend Θ_0' by adding $\Diamond j$ to the current l-block. This was what was required for this subcase.

Subcase iv). Suppose Θ_0 is extended by Nom having as input $\neg @_k\psi$, which descends from $\Diamond k$ on B_3. This case is precisely as the previous subcase. Here it's just $\neg @_k\psi$ instead of $\Diamond k$. Both descend from $\Diamond k$ in B_3. □

Lemma 4.3 (Smullyan-Fitting block lemma for `ST`) *If S is a maximal* `ST`*-consistent set of finite named blocks, then S is a Hintikka set.*

Proof. Given a maximal $\mathtt{ST}$-consistent set S of finite named blocks, we can prove that all Hintikka properties hold for S. Here we give the crucial step for property vi, which is where Bridge was used in the proof for $\mathtt{STB}$.

Property vi. Suppose there is an i-block $B_1' \in S$ with $\Diamond j$ on it and also a j-block $B_2' \in S$ with k on it. Suppose there is no i-block with $\Diamond k$ on it in S. Let B_3 be the i-block consisting just of i and $\Diamond k$. By maximality of S we have that $S \cup \{B_3\}$ is inconsistent. Thus, for a finite $S^f \subseteq S$ there is a closed tableau T for $S^f \cup \{B_3\}$. The procedure given in the proof of the elimination lemma converts T into a tableau T' having S^f, B_1 and B_2 as root blocks, where B_1 and B_2 are as in the elimination lemma. We now show that T' can be turned into a closed tableau (in case it isn't one already), which implies that there exists a closed tableau having S^f, B_1' and B_2' as root blocks.

Take any branch Θ of T. As Θ is closed there exist l-blocks B_4 and B_5 such that $\varphi \in B_4$ and $\neg\varphi \in B_5$. Either φ or $\neg\varphi$ descends from $\Diamond k$ on B_3 in T or neither of them does. If neither φ or $\neg\varphi$ descends from $\Diamond k$ on B_3, then by the elimination lemma there are l-blocks B_4' and B_5' occurring on Θ' such that φ and $\neg\varphi$ occur on B_4' and B_5', respectively. But then Θ' is closed.

Suppose therefore, that φ descends from $\Diamond k \in B_3$. Then we have two subcases: φ is either $\Diamond k$ or φ is $\neg @_k\psi$.

Subcase 1: φ is $\Diamond k$. According to the elimination lemma, we here have a branch Θ' of T' containing an l-block with $\Diamond j$ on it (as $\Diamond j$ is the j-replacement of $\Diamond k$) and an l-block with $\neg\Diamond k$ (note that $\neg\Diamond k$ cannot descend from $\Diamond k$). We extend Θ' by an application of GoTo opening an l-block. Then we apply $(\neg\Diamond)$ on input $\Diamond j$ and $\neg\Diamond k$ and retrieve $\neg @_j k$. Then open a j-block with GoTo and after that we apply $(\neg @)$ in order to add $\neg k$ to the current j-block. This extension of Θ' is closed, as k occurs on the j-block B_2.

Subcase 2: φ is $\neg @_k\psi$. In this case we have on Θ' an l-block with $\neg @_j\psi$ and an l-block with $\neg\neg @_k\psi$. We can extend Θ' by an application of GoTo, thereby opening an l-block. Then we apply $(\neg\neg)$ in order to add $@_k\psi$ to the current l-block. Then we open an j-block with GoTo, and by applying $(\neg @)$ we retrieve $\neg\psi$, by applying Nom we retrieve k since k occurs in the j-block B_2, and finally, we retrieve ψ by an application of $(@)$. This extension of Θ' is closed.

The case where $\neg\varphi$ descends from $\Diamond k$ in B_3 is similar. Then we have an l-block on Θ' with $@_k\psi$ and an l-block with $\neg @_j\psi$. We proceed as before. □

Theorem 4.4 *Every* $\mathtt{ST}$*-consistent formula is satisfiable on a named model.*

Proof. Like the proof of Theorem 3.6, but making use of the Smullyan-Fitting block lemma for $\mathtt{ST}$. □

5 Extensions of Basic Hybrid Logic

We now discuss Seligman-style systems for various extended hybrid logics.

5.1 Multimodal hybrid logic

Instead of working with a single $\Diamond$ we could have a collection of diamonds $\langle r \rangle$, where $r \in R$ for some (typically finite) index set R. However (assuming each modality is independent) there is little to say here: we simply add a version of the diamond (and box) rule for each additional modality. The completeness proofs generalize: the Hintikka properties are duplicated for each modality, the induced model is built as in the unimodal case, and the Lindenbaum-Henkin construction proceeds as before, but with witnessing blocks for *all* diamonds.

5.2 Pure axioms

A hybrid formula ρ is pure if it contains no ordinary propositional symbols: for example, $i \to \neg\Diamond i$, $\Diamond\Diamond i \to \Diamond i$ and $@_i\Box(\Diamond i \to i)$ are pure. Every pure formula defines a first-order definable class of frames; the examples just given define the classes of irreflexive, transitive, and antisymmetric frames respectively. [7] Moreoever, as we now show, when pure formulas are used as axioms, the resulting system is complete with respect to the class of frames the axioms define (the class of frames on which every axiom is valid). [8]

Let $i_1, \ldots, i_n$ be the nominals in ρ; then $\rho(j_1, \ldots, j_n/i_1, \ldots, i_n)$ is the formula obtained when we uniformly substitute $j_1, \ldots, j_n$ for $i_1, \ldots, i_n$ in ρ. We call $\rho(j_1, \ldots, j_n/i_1, \ldots, i_n)$ a *pure instance* of ρ. Let Axiom be some set of pure formulas. The rules of ST-Axiom are the rules of ST together with the rule that if $\rho \in \mathsf{Axiom}$, then we can add any pure instance of ρ to the current block.

Theorem 5.1 **(Axiom-completeness)** *Every ST-Axiom-consistent formula is satisfiable on a named model based on a frame that belongs to the class that Axiom defines.*

Proof. Let φ be the ST-Axiom-consistent formula. Take it as the root formula of a tableau, immediately apply rule Name to ensure that the initial block is named, and use the Lindenbaum-Henkin construction to form a maximal ST-Axiom-consistent S of finite named blocks. Assume for the sake of contradiction that the induced model $\mathfrak{M} = (\mathfrak{F}, V)$ is based on frame $\mathfrak{F}$ *not* belonging to the class defined by Axiom, that is, $\mathfrak{F} \not\models \rho$ for some $\rho \in \mathsf{Axiom}$. This means, that for some valuation function V' and some world w we have

$$(\mathfrak{F}, V'), w \not\models \rho.$$

Let $i_1, \ldots, i_n$ be the nominals of ρ, and suppose $V'(i_1) = \{w_1\}, \ldots, V'(i_n) = \{w_n\}$. And now the usefulness of named models becomes clear: because $\mathfrak{M}$ is named, there are nominals $i, j_1, \ldots, j_n$ such that $|i| = w$ and $V(j_1) = \{w_1\}, \ldots, V(j_n) = \{w_n\}$. Uniformly substituting $j_1 \ldots, j_n$ for $i_1, \ldots, i_n$ yields:

$$\mathfrak{M}, |i| \not\models \rho(j_1 \ldots, j_n/i_1 \ldots i_n).$$

[7] A formula φ defines a class of frames F when $\mathfrak{F} \models \varphi$ iff $\mathfrak{F}$ belongs to F. Here $\mathfrak{F} \models \varphi$ (φ is valid on $\mathfrak{F}$) means that φ is true at every world in $\mathfrak{F}$ under any valuation.

[8] Such results are familiar to hybrid logicians: see in particular [3], [5] and [11]; the first reference contains the result most closely related to the result we shall now prove.

The Hintikka block lemma implies that no i-block with $\rho(j_1,\ldots,j_n/i_1,\ldots,i_n)$ on is an element of S. Therefore, by maximal ST-Axiom-consistency of S adding the i-block B with just i and $\rho(j_1,\ldots,j_n/i_1,\ldots,i_n)$ on makes it ST-Axiom-inconsistent. So there exists a closed ST-Axiom-tableau T for $\{B\}\cup S^f$, where S^f is some finite subset of S. Now use S^f as the initial blocks of a new tableau. We assume i occurs in S^f (otherwise we just add a block from S containing i), so below S^f we open an i-block by GoTo. To this i-block we add the axiom $\rho(j_1\ldots,j_n/i_1\ldots i_n)$. Below this we can (modulo renaming of fresh nominals) paste our closed tableau T. But this contradicts the ST-Axiom-consistency of S as $S^f\subseteq S$, so $\mathfrak{F}$ does in fact belong to the class defined by Axiom. □

5.3 Hybrid tense logic

The best known multimodal logic with linked diamonds is probably tense logic. The diamond pair F and P both make use of the same relation: F looks forward along it (towards the future), and P looks backward (towards the past).

$$\mathfrak{M},w\models\mathsf{F}\varphi \text{ iff for some } w',\ wRw' \text{ and } \mathfrak{M},w'\models\varphi$$
$$\mathfrak{M},w\models\mathsf{P}\varphi \text{ iff for some } w',\ w'Rw \text{ and } \mathfrak{M},w'\models\varphi.$$

We call such models *bidirectional*. The box-form $\mathsf{G}\varphi$ means φ holds at all times in the future, and $\mathsf{H}\varphi$ means φ holds at all times in the past. The tableau rules for these operators are obtained by instantiating the generic diamond rules given in figure 2 by F and P. As F and P explore the same relation in opposite directions, we *also* add the transposition rules in figure 6 thereby obtaining the calculus $\mathsf{ST}_{(\mathsf{P},\mathsf{F})}$.

Theorem 5.2 ((P,F)-completeness) *Every $\mathsf{ST}_{(\mathsf{P},\mathsf{F})}$-consistent formula is satisfiable on a named bidirectional model.*

Proof. Let φ be the $\mathsf{ST}_{(\mathsf{P},\mathsf{F})}$-consistent formula which via the Lindenbaum-Henkin construction leads to $\mathfrak{M}=(T,R_\mathsf{F},R_\mathsf{P},V)$ as the induced named model. We want to show for all $t,t'\in T$ that $tR_\mathsf{F}t'$ iff $t'R_\mathsf{P}t$. Suppose for the sake of contradiction that this is not the case: either we have $tR_\mathsf{F}t'$ but not $t'R_\mathsf{P}t$, or we have $t'R_\mathsf{P}t$ but not $tR_\mathsf{F}t'$. We prove (by using F-trans) the first leads to a contradiction; the second is completely analogous (using P-trans instead) So assume $tR_\mathsf{F}t'$ but not $t'R_\mathsf{P}t$. As all elements of T are equivalence classes of nominals, this means that there are nominals i and j such that $|i|R_\mathsf{F}|j|$ but not $|j|R_\mathsf{P}|i|$. As it is not the case that $|j|R_\mathsf{P}|i|$, and as $\mathfrak{M},|i|\models i$, it follows that $\mathfrak{M},|j|\not\models\mathsf{P}i$. So by the construction of $\mathfrak{M}$ there cannot be any j-block in S with $\mathsf{P}i$ on it. But now we have a problem: as $|i|R_\mathsf{F}|j|$, there is there is an i-block with $\mathsf{F}j$ on it. In this block we can apply GoTo to open a j-block and we can then apply F-trans to extend it with $\mathsf{P}i$—in short, we have built a j-block in S with $\mathsf{P}i$ on it: contradiction. We conclude that the frame is bidirectional. □

5.4 The universal modality

The universal modality is a standard tool in hybrid logic. Formulas of the form $\mathsf{E}\varphi$ are satisfied at a world w in a model if there is some world in the model

$$\begin{array}{c} i \\ \mathsf{F}j \\ \hline \vdots \\ j \\ \Big| \ (\mathsf{F\text{-}trans}) \\ \mathsf{P}i \end{array} \qquad\qquad \begin{array}{c} i \\ \mathsf{P}j \\ \hline \vdots \\ j \\ \Big| \ (\mathsf{P\text{-}trans}) \\ \mathsf{F}i \end{array}$$

Fig. 6. Transposition rules for hybrid tense logic.

that satisfies φ. That is:

$$\mathfrak{M}, w \models \mathsf{E}\varphi \text{ iff for some } w' \text{ we have } \mathfrak{M}, w' \models \varphi.$$

The dual of E is A. Formulas of the form $\mathsf{A}\varphi$ are satisfied at world w in a model if all worlds in the model satisfy φ. In short, the universal modality uses the universal relation $W \times W$ on worlds. [9] We capture its logic by adding the generic diamond rules plus the rule in figure 7, thereby obtaining the $\mathtt{ST}_{\mathsf{E}}$ calculus. For an example of the rules at work see figure 8.

$$\begin{array}{c} \Big| \ \ \mathsf{Universal}^1 \\ \mathsf{E}i \end{array}$$

[1] The nominal i must already occur on the branch.

Fig. 7. The rule for the universal modality.

Theorem 5.3 (E-completeness) *Every $\mathtt{ST}_{\mathsf{E}}$-consistent formula is satisfiable on a named model $\mathfrak{M}$ with R_{E} being $W \times W$.*

Proof. Let S be the maximal $\mathtt{ST}_{\mathsf{E}}$-consistent set produced by the $\mathtt{ST}_{\mathsf{E}}$-consistent φ inducing the model $\mathfrak{M}$ which has R_{E} as the accessibility relation for E. Assume conversely that E is not universal. By construction of the worlds in W there exists nominals i and j, such that both i and j occur as names for blocks in S, and it is not the case that $|i|R_{\mathsf{E}}|j|$. By the definition of R_{E} this means that there are no i-blocks in S with $\mathsf{E}j$ on them. So, take the i-block with just $\mathsf{E}j$ and i on it and call it B. It follows from the maximality of S

[9] So the universal modality is just as $\mathsf{S5}$ operator. But it is usually used as a tool to express global constraints involving other modalities, for example: $A(\Box p \to \Diamond q)$. Note that both $\mathsf{E}(i \wedge \varphi)$ and $\mathsf{A}(i \to \varphi)$ are ways of expressing $@_i\varphi$.

1	$\neg(\Diamond\varphi \to \mathsf{E}\varphi)$	
2	$\Diamond\varphi$	$(\neg\to)$ on 1
3	$\neg\mathsf{E}\varphi$	$(\neg\to)$ on 1
4	$\Diamond i$	$(\Diamond)$ on 2
5	$@_i\varphi$	$(\Diamond)$ on 2
6	$\mathsf{E}i$	Universal
7	$\neg @_i\varphi$	(E) on 3 and 6
	$\times$	

1	$\neg(\mathsf{E}\varphi \to \varphi \vee \mathsf{D}\varphi)$	
2	$\mathsf{E}\varphi$	$(\neg\to)$ on 1
3	$\neg(\varphi \vee \mathsf{D}\varphi)$	$(\neg\to)$ on 1
4	$\neg\varphi$	$(\neg\vee)$ on 3
5	$\neg\mathsf{D}\varphi$	$(\neg\vee)$ on 3
6	$\mathsf{E}i$	(E) on 2
7	$@_i\varphi$	(E) on 2

8	i		9	$\neg i$	EM-nom
10	φ	(@)	11	$\mathsf{D}i$	D_1 on 9
	$\times$		12	$\neg @_i\varphi$	$(\neg\mathsf{D})$ on 5,11
				$\times$	

Fig. 8. Two examples of the rules at work. On the left we see a way of using the rules for $\Diamond$ and E to prove $\Diamond\varphi \to \mathsf{E}\varphi$: if φ is true at an *accessible* world, then it is true at *some* world in the model. The example on the right shows how to prove $\mathsf{E}\varphi \to \varphi \vee \mathsf{D}\varphi$ using the rules for D and E.

that $S \cup \{B\}$ is $\mathtt{ST}_\mathsf{E}$-inconsistent. Therefore, there exists finite $S^f \subseteq S$, such that $S^f \cup \{B\}$ has a closed tableau. Let B_i and B_j be blocks in S containing i and j, respectively; such blocks exist by our initial assumption. Then we can construct a closed $\mathtt{ST}_\mathsf{E}$-tableau from $S^f \cup \{B_i\} \cup \{B_j\}$, since below these initial blocks we can apply GoTo and open an i-block and then apply Universal to add $\mathsf{E}j$. Below this we simply paste (modulus renaming of fresh variables) the closed tableau we are given for $S^f \cup \{B\}$. This, however, contradicts the $\mathtt{ST}_\mathsf{E}$-consistency of S, as $S^f \cup \{B_i\} \cup \{B_j\}$ without loss of generality can be assumed to be a subset of S □

5.5 The difference operator

The satisfaction definition for D is:

$$\mathfrak{M}, w \models \mathsf{D}\varphi \text{ iff there exists } w' \neq w \text{ such that } \mathfrak{M}, w' \models \varphi.$$

That is, D is evaluated using the difference relation ($W \times W$ minus the diagonal-elements).[10] We get a complete system $\mathtt{ST}_\mathsf{D}$ by taking the usual diamond rules together with the rules shown figure 9. For an illustrative example we derive in the calculus $\mathsf{E}\varphi \to \varphi \vee \mathsf{D}\varphi$ using both the E- and the D-rules, see figure 8. These extra rules force R_D to be the difference relation.

Theorem 5.4 (D**-completeness)** *Every $\mathtt{ST}_\mathsf{D}$-consistent formula is satisfiable on a named model $\mathfrak{M}$, where R_D is the difference relation.*

Proof. Suppose an $\mathtt{ST}_\mathsf{D}$-consistent formula leads to the maximal $\mathtt{ST}_\mathsf{D}$-consistent set S, which induces a model $\mathfrak{M}$ having relation R_D. Suppose R_D isn't the difference relation. Then (i) there is a world from W related by R_D to itself or (ii) there are distinct worlds in W that are not related by R_D. By construction of the set of world it follows that either:

[10] The difference operator is stronger than the universal modality. $\mathsf{E}\varphi$ can be defined to be $\varphi \vee \mathsf{D}\varphi$, but D cannot be defined in terms of E; see [11] for detailed discussion.

$$
\begin{array}{ccc}
\begin{array}{c} \cdot \\ \swarrow \quad \searrow \ \ \textsf{EM-nominal}^{*} \\ i \qquad\quad \neg i \end{array}
&
\begin{array}{c} \neg i \\ \Big| \ (\mathsf{D}_1) \\ \mathsf{D}i \end{array}
&
\begin{array}{c} \mathsf{D}i \\ \Big| \ (\mathsf{D}_2) \\ \neg i \end{array}
\end{array}
$$

* The nominal i must already occur on the branch.

Fig. 9. Rules for the difference operator.

(i) There is a nominal i occuring in S such that $|i|R_{\mathsf{D}}|i|$, or

(ii) There are nominals i and j occuring in S such that $|i| \neq |j|$ but not $|i|R_{\mathsf{D}}|j|$.

The first case is easy: $|i|R_{\mathsf{D}}|i|$ means that there is an i-block B in S with $\mathsf{D}i$ on it. We can immediately construct a closed tableau from B by applying the rule D_2 to $\mathsf{D}i$ to get $\neg i$, which contradicts the consistency of S.

Now for the second case. As $|i| \neq |j|$ there can be no i-block with j on it in S. Let B_1 be the i-block with just i and j on it. As S is maximal $\mathtt{ST}_{\mathsf{D}}$-consistent there exists a finite $S_1^f \subseteq S$ and a closed tableau T_1 for $\{B_1\} \cup S_1^f$. Moreover, not $|i|R_{\mathsf{D}}|j|$ being the case implies (by definition of $|i|R_{\mathsf{D}}|j|$) that there cannot be any i-block with $\mathsf{D}j$ on in S. So if B_2 is the i-block with just i and $\mathsf{D}j$ on it, then (by the maximality of S) there exists a finite $S_2^{\mathsf{f}} \subseteq S$ and a closed tableau T_2 for $S_2^f \cup \{B_2\}$. On the other hand, we can construct the following $\mathtt{ST}_{\mathsf{D}}$-tableau (if i or j does not occur in $S_1^f \cup S_2^f$ we just add blocks from S with these as appropriate; such blocks clearly exist):

$$
\begin{array}{c}
S_1^f \\
S_2^f \\ \hline
i \\
\swarrow \qquad \searrow \\
j \qquad\qquad \begin{array}{ll} \neg j & \textsf{EM-nominal} \\ \mathsf{D}j & (\mathsf{D}_1) \end{array}
\end{array}
$$

Below the left-hand branch we can paste the closed tableau T_1, and below the right-hand branch, closed tableau T_2 (modulus renaming of fresh variables). As $S_1^f \cup S_2^f$ is a subset of the $\mathtt{ST}_{\mathsf{D}}$-consistent set S, no such tableau exists. $\Box$

6 Concluding remarks

The Seligman-style approach to hybrid inference is an intriguing alternative to the better-known labelling methods, and in this paper we have presented some first results on extending Seligman-style tableau inference to a wider ranger of hybrid logics and languages. Much remains to be done. For a start, we would like to prove similar results for the terminating tableau system $\mathtt{ST}^{\star}$ discussed in [2], and, if possible, to try and show that Nom can be restricted in its application to nominals and near-atomic formulas of the form $\Diamond i$. In addition, we would like to develop Seligman-style tableau systems for various versions of first-order hybrid logic (with the downarrow binder $\downarrow$). This is a useful language for rethinking some traditional issues in philosophical logic (scope, equality, indexicality, actuality, existence and definite descriptions) and we believe Seligman-style tableaus are the simplest tools for working with it.

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