Time Series Analysis

Solutions to problems in Chapter 2

IMM

Solution 2.1

Question 1.

A second order momement representation of $(H \quad L)^\top$ consists of

$$E[H], \quad V[H], \quad \rho[H,L], \quad E[L], \quad V[L]$$

which are calculated as

$$\begin{aligned}
E[H] &= E[2X+3Y] = 2E[X]+3E[Y] = \underline{\underline{40}} \\
E[L] &= E[-X+2Y] = -E[X]+2E[Y] = \underline{\underline{15}} \\
V[H] &= V[2X+3Y] = 2^2V[X]+3^2V[Y]+2\cdot 6\mathrm{Cov}[X,Y] \\
&= 2^2\cdot 1+3^2\cdot 2^2+2\cdot 6\cdot 1\cdot 2\cdot\frac{1}{2} = \underline{\underline{52}} \\
V[L] &= V[-X+2Y] = (-1)^2V[X]+2^2V[Y]-2\cdot 2\cdot\mathrm{Cov}[X,Y] \\
&= (-1)^2\cdot 1+2^2\cdot 2^2-2\cdot 2\cdot 1\cdot 2\cdot\frac{1}{2} = \underline{\underline{13}} \\
\mathrm{Cov}[H,L] &= \mathrm{Cov}[2X+4Y,-X+2Y] \\
&= -2V[X]+6V[Y]+4\mathrm{Cov}[X,Y]-3\mathrm{Cov}[X,Y] \\
&= -2\cdot 1+6\cdot 2^2+1\cdot 1\cdot 2\cdot\frac{1}{2} = 23 \quad\Rightarrow \\
\rho[H,L] &= \frac{\mathrm{Cov}[H,L]}{\sqrt{V[X]V[Y]}} = \underline{\underline{\frac{23}{26}}}
\end{aligned}$$

Alternatively, we define

$$B = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

I.e.

$$\begin{bmatrix} H \\ L \end{bmatrix} = B\begin{bmatrix} X \\ Y \end{bmatrix}$$

and the second order moment representation of $(H \quad L)^\top$ can then be calculated by

$$
\begin{aligned}
E\left[\left[\begin{array}{c} H \\ L \end{array}\right]\right] &= BE\left[\left[\begin{array}{c} X \\ Y \end{array}\right]\right] = \left[\begin{array}{rr} 2 & 3 \\ -1 & 2 \end{array}\right]\left[\begin{array}{c} 5 \\ 10 \end{array}\right] = \underline{\underline{\left[\begin{array}{c} 40 \\ 15 \end{array}\right]}} \\
V\left[\left[\begin{array}{c} H \\ L \end{array}\right]\right] &= BV\left[\left[\begin{array}{c} X \\ Y \end{array}\right]\right]B^\top = \left[\begin{array}{rr} 2 & 3 \\ -1 & 2 \end{array}\right]\left[\begin{array}{cc} 1 & 1 \\ 1 & 4 \end{array}\right]\left[\begin{array}{rr} 2 & -1 \\ 3 & 2 \end{array}\right] = \\
&= \underline{\underline{\left[\begin{array}{cc} 52 & 23 \\ 23 & 13 \end{array}\right]}}
\end{aligned}
$$

In a second order moment representation the connection between the two stochastic variables can either be representated by the correlation $\rho[X, Y]$ or by the covariance $\mathrm{Cov}[X, Y]$.

Solution 2.2

Question 1.

$$E[Y|X] = E[\alpha + \beta X + \epsilon|X] = \alpha + \beta E[X|X] + E[\epsilon|X] = \underline{\underline{\alpha + \beta X}}$$

$$V[Y|X] = V[\alpha + \beta X + \epsilon|X] = \beta^2 V[X|X] + V[\epsilon|X] = \underline{\underline{\sigma_\epsilon^2}}$$

(Since $V[X|X] = E[(X - E[X|X])^2|X] = 0$)

Question 2.

$$\begin{aligned} E[Y] &= E[E[Y|X]] = E[\alpha + \beta X] = \alpha + \beta E[X] = \underline{\underline{\alpha + \beta \mu_x}} \\ V[Y] &= E[V[Y|X]] + V[E[Y|X]] = E[\sigma_\epsilon^2] + V[\alpha + \beta X] = \underline{\underline{\sigma_\epsilon^2 + \beta^2 \sigma_X^2}} \end{aligned}$$

Question 3.

(We define $\mu_Y = E[Y]$, $\sigma_Y^2 = V[Y]$ and $\sigma_X^2 = V[Y]$)

$$\text{Cov}[X, Y] = \text{Cov}[X, \alpha + \beta X + \epsilon] = \beta \sigma_X^2$$

And since $\text{Cov}[X, Y] = \rho(X, Y)\sigma_X \sigma_Y = \rho \sigma_X \sigma_Y$ the moment estimate of β is

$$\beta = \underline{\underline{\rho \frac{\sigma_Y}{\sigma_X}}}$$

From the first result in question 2 α is found as

$$\alpha = \mu_Y - \beta \mu_x = \underline{\underline{\mu_Y - \frac{\rho \sigma_Y \mu_X}{\sigma_X}}}$$

Solution 2.3

Question 1.

The mean and variance of Y can be found by first determining the conditional mean and variance of Y given N. The conditional mean is

$$E[Y|N] = E[X_1 + X_2 + \cdots + X_N|N] = N \cdot E[X_i] \ ,$$

and the conditional variance is

$$V[Y|N] = V[X_1 + X_2 + \cdots + X_N|N] = N \cdot V[X_i]$$

applying for the conditional variance that $\mathrm{Cov}[X_r, X_s|N = j] = 0$ for $r \neq s$. The mean is obtained by using the property (2.43)

$$\begin{aligned} E[Y] &= E[E[Y|N]] = E[N \cdot E[X_i]] = E[N] \cdot E[X_i] \\ &= 20 \cdot 2 = \underline{\underline{40}} \ , \end{aligned}$$

and by using the theorem (2.51), we get

$$\begin{aligned} V[Y] &= E[V[Y|N]] + V[E[Y|N]] \\ &= E[N \cdot V[X_i]] + V[N \cdot [E[X_i]] \\ &= E[N] \cdot V[X_i] + E[X_i]^2 \cdot V[N] \\ &= 20 \cdot (\frac{1}{8})^2 + 2^2 \cdot 2^2 = \underline{\underline{\frac{261}{16}}} \ . \end{aligned}$$

Question 2.

By using the theorem (2.52)

$$\mathrm{Cov}[Y, Z] = E[\mathrm{Cov}[Y, Z|N]] + \mathrm{Cov}[E[Y|N], E[Z|N]] \ ,$$

where $E[\text{Cov}[Y,Z|N]] = E[N \cdot V[X_i]]$ as $\text{Cov}[X_r, X_s|N] = 0$ for $r \neq s$, i.e.

$$\text{Cov}[Y,Z] = E[N \cdot V[X_i]] + \text{Cov}[N \cdot E[X_i], N \cdot \alpha E[X_i]]$$

$$= E[N] \cdot V[X_i] + E[X_i]^2 \alpha \cdot V[N]$$

$$= 20 \cdot (\frac{1}{8})^2 + 2^2 \cdot \alpha \cdot 2^2$$

$$= \frac{5}{16} + 16\alpha = \underline{\underline{\frac{5+256\alpha}{16}}} \, .$$