Introduction to General and Generalized Linear Models

Hierarchical models

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April 29, 2012

This lecture

Takes a deeper look into the use of random effects instead of overdispersion, which we have already used in a few examples.

- Introduction, approaches to modelling of overdispersion
- Hierarchical Poisson Gamma model
- Bayesian detour
- The Binomial Beta model
- Normal distributions with random variance
- Hierarchical generalized linear models

Introduction

- A characteristic property of the generalized linear models is that the variance, $\mathrm{Var}[Y]$ in the distribution of the response is a known function, $V(\mu)$, that only depends on the mean value μ

$$\mathrm{Var}[Y_i] = \lambda_i V(\mu) = \frac{\sigma^2}{w_i} V(\mu)$$

  where w_i denotes a known *weight*, associated with the i'th observation, and where σ^2 denotes a *dispersion parameter* common to all observations, irrespective of their mean.

- The dispersion parameter σ^2 does serve to express *overdispersion* in situations where the residual deviance is larger than what can be attributed to the variance function $V(\mu)$ and known weights w_i.

- We shall describe an alternative method for modeling overdispersion, viz. by *hierarchical models* analogous to the mixed effects models for the normally distributed observations.

Introduction

- A starting point in a hierarchical modeling is an assumption that the distribution of the random "noise" may be modeled by an exponential dispersion family (Binomial, Poisson, etc.), and then it is a matter of choosing a suitable (prior) distribution of the mean-value parameter μ.
- It seems natural to choose a distribution with a support that coincides with the mean value space $\mathcal{M}$ rather than using a normal distribution (with a support constituting all of the real axis $\mathbb{R}$).
- In some applications an approach with a normal distribution of the canonical parameter is used. Such an approach is sometimes called *generalized linear mixed models* (GLMMs)

Introduction

- Although such an approach is consistent with a formal requirement of equivalence between mean values space and support for the distribution of μ in the binomial and the Poisson distribution case, the resulting marginal distribution of the observation is seldom tractable, and the likelihood of such a model will involve an integral which cannot in general be computed explicitly.
- Also, the canonical parameter does not have a simple physical interpretation and, therefore, an additive "true value" + error, with a normally distributed "error" on the canonical parameter to describe variation between subgroups, is not very transparent.
- Instead, we shall describe an approach based on the so-called *standard conjugated distribution* for the mean parameter of the within group distribution for exponential families.
- These distributions combine with the exponential families in a simple way, and lead to marginal distributions that may be expressed in a closed form suited for likelihood calculations.

Hierarchical Poisson Gamma model - example

The table shows the distribution of the number of daily episodes of thunderstorms at Cape Kennedy, Florida, during the months of June, July and August for the 10-year period 1957–1966, total 920 days.

Number of episodes, z_i	Number of days, $\# \ i$	Poisson expected
0	803	791.85
1	100	118.78
2	14	8.91
3+	3	0.46

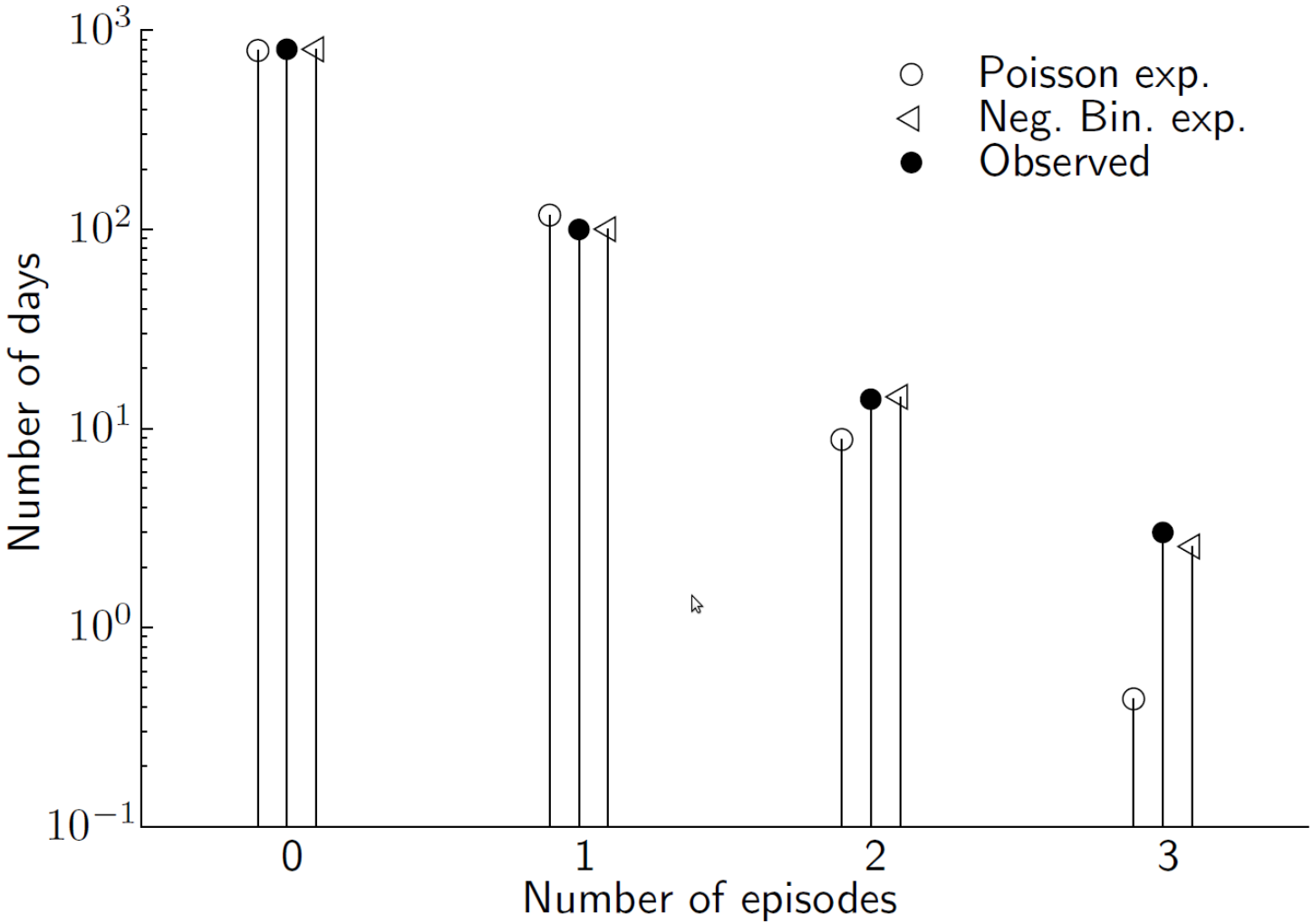
Table: The distribution of days with 0, 1, 2 or more episodes of thunderstorm at Cape Kennedy.

All observational periods are $n_i = 1$ day.

Hierarchical Poisson Gamma model - example

- The data represents *counts* of events (episodes of thunderstorms) distributed in time.
- A completely random distribution of the events would result in a Poisson distribution of the number of daily events.
- The variance function for the Poisson distribution is $V(\mu) = \mu$; therefore, a Poisson distribution of the daily number of events would result in the variance in the distribution of the daily number of events being equal to the mean, $\widehat{\mu} = \overline{y}_+ = 0.15$ thunderstorms per day.
- The empirical variance is $s^2 = 0.1769$, which is somewhat larger than the average. We further note that the observed distribution has *heavier tails* than the Poisson distribution. Thus, one might be suspicious of overdispersion.

Hierarchical Poisson Gamma model - example



Formulation of hierarchical model

Theorem (Compound Poisson Gamma model)

Consider a hierarchical model for Y specified by

$$Y|\mu \sim Pois(\mu),$$
$$\mu \sim G(\alpha, \beta),$$

i.e. a two stage model.

In the first stage a random mean value μ is selected according to a Gamma distribution, and that Y is generated according to a Poisson distribution with that value as mean value. Then the marginal distribution of Y is a negative binomial distribution, $Y \sim \mathrm{NB}(\alpha, 1/(1+\beta))$

Formulation of hierarchical model

Theorem (Compound Poisson Gamma model, continued)

The probability function for Y is

$$\begin{aligned} P[Y = y] &= g_Y(y; \alpha, \beta) \\ &= \frac{\Gamma(y+\alpha)}{y!\Gamma(\alpha)} \frac{\beta^y}{(\beta+1)^{y+\alpha}} \\ &= \binom{y+\alpha-1}{y} \frac{1}{(\beta+1)^\alpha} \left(\frac{\beta}{\beta+1}\right)^y \quad \text{for } y = 0, 1, 2, \ldots \end{aligned}$$

where we have used the convention

$$\binom{z}{y} = \frac{\Gamma(z+1)}{\Gamma(z+1-y)\ y!}$$

for z real and y integer values.

Proof.

We have the two densities:

$$f_{Y|\mu}(y) = \frac{\mu^y}{y!}e^{-\mu} \quad \text{and} \quad f_\mu(\mu,\alpha,\beta) = \frac{1}{\beta^\alpha\Gamma(\alpha)}\mu^{\alpha-1}e^{-\frac{\mu}{\beta}}$$

$$g_Y(y) = \int_0^\infty \frac{\mu^y}{y!}e^{-\mu}\frac{1}{\beta^\alpha\Gamma(\alpha)}\mu^{\alpha-1}e^{-\frac{\mu}{\beta}}\,d\mu \quad \text{[collect, and constants outside]}$$

$$= \frac{1}{y!\beta^\alpha\Gamma(\alpha)}\int_0^\infty \mu^{\overbrace{y+\alpha}^{\widetilde{\alpha}}-1}e^{-\mu\overbrace{\left(\frac{(\beta+1)}{\beta}\right)}^{1/\widetilde{\beta}}}\,d\mu \quad \text{[recognize as } \Gamma \text{ integral]}$$

$$= \frac{1}{y!\beta^\alpha\Gamma(\alpha)}\frac{\Gamma(y+\alpha)\left(\frac{\beta}{(\beta+1)}\right)^{y+\alpha}}{1} \quad \text{[reduce]}$$

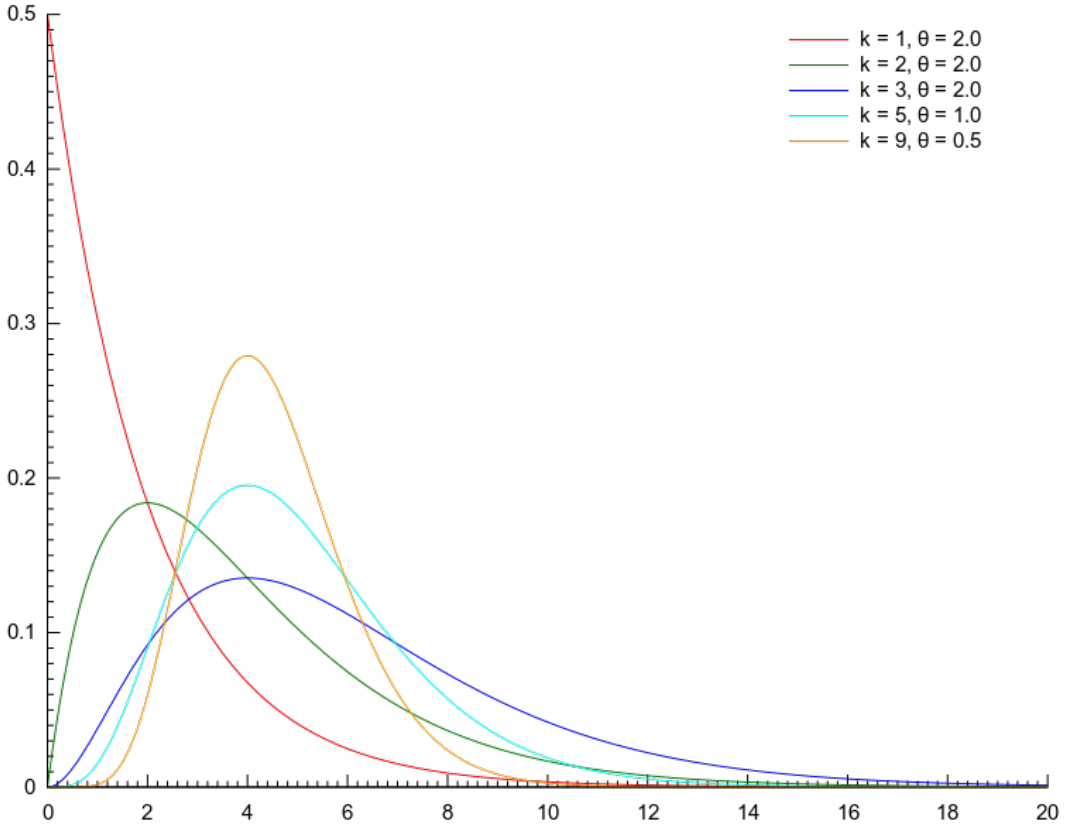
$$= \frac{\Gamma(y+\alpha)\beta^y}{y!\Gamma(\alpha)(\beta+1)^{y+\alpha}} \quad \text{[done!]}$$

Formulation of hierarchical model

- For integer values of α the negative binomial distribution is known as the distribution of the number of "failures" until the α'th success in a sequence of independent Bernoulli trials where the probability of success in each trial is $p = 1/(1+\beta)$.
- For $\alpha = 1$ the distribution is known as the *geometric distribution*.

Why use a Gamma to describe variation between days?

- It has the desired support
- It is a very flexible distribution



- Last but not least the integral can be directly calculated.

Inference on mean μ

Theorem (Conditional distribution of μ)

Consider the hierarchical Poisson-Gamma model and assume that a value $Y = y$ has been observed.

Then the conditional distribution of μ for given $Y = y$ is a Gamma distribution,

$$\mu|\ Y = y\ \sim G(\alpha + y, \beta/(\beta + 1))$$

with mean

$$\mathrm{E}[\mu|\ Y = y] = \frac{\alpha + y}{(1/\beta + 1)}$$

Proof is: 1. Bayes' theorem, 2. Collect terms, 3. Recognize Gamma

Back to the thunder storm example

The data was:

Number of episodes, z_i	Number of days, $\# \ i$	Poisson expected
0	803	791.85
1	100	118.78
2	14	8.91
3+	3	0.46

- Notice that the observations have been summarized for us
- The real data would be something like:

Day	Number of storms
1	0
2	0
3	1
.	.
.	.
.	.
920	0

- The model we want to setup is fairly simple:

$$Y_i \sim NB(\alpha, 1/(1+\beta)), \quad \text{where } i = 1...920.$$

- As the observations are collected, so can we collect the likelihood calculations

$$803 \cdot \ell(0) + 100 \cdot \ell(1) + 14 \cdot \ell(2) + 3 \cdot \ell(\geq 3)$$

- Remember that:

$$P(Y \geq 3) = 1 - P(Y = 0) - P(Y = 1) - P(Y = 2)$$

Detour: Bayesian inference

- Purely likelihood based inference (a.k.a. Frequentist inference) is based on drawing information from data Y about the model parameters θ via the likelihood function:

$$L(Y|\theta)$$

- In Bayesian inference prior beliefs about the model parameters are expressed as a probability density, so we have:

$$L(Y|\theta) \qquad \text{and} \qquad q(\theta|\psi)$$

- Inference about the model parameters are drawn from the posterior density:

$$p(\theta|Y=y) = \frac{L(Y=y|\theta)\,q(\theta|\psi)}{\int L(Y=y|\theta)\,q(\theta|\psi)\,d\theta}$$

which is computed via Bayes' rule.

Detour: Bayesian inference

- What is done here is to update the prior beliefs with data
- If the data part is dominating results close to likelihood inference can be expected
- Notice that the prior parameters ψ are not influenced by data. In hierarchical/mixed/random effects models we would estimate those.
- Notice that the prior assumption is entirely subjective and not subject to model validation. In hierarchical/mixed/random effects models we can - to some extend - validate our assumed distribution.

Detour: Bayesian inference

- Notice that the integral in the posterior denominator in general cannot be calculate analytically.
- Before the widespread use of MCMC* it was very important to specify priors such that the denominator integral could be calculated.
- A prior density is said to be conjugated to a certain likelihood if the posterior density has the same parametric form as the prior density.
- Using conjugate priors simplifies the modeling. To derive the posterior distribution, it is not necessary to perform the integration, as the posterior distribution is simply obtained by updating the parameters of the prior one.

*Markov Chain Monte Carlo methods are simulations techniques that allow you to sample a Markov chain with a desired equilibrium density, when that density is only know unnormalized

Reparameterization of the Gamma distribution

Instead of the usual parameterization of the gamma distribution of μ by its shape parameter α and scale parameter β, we may choose a parameterization by the mean value, $m = \alpha\beta$, and the signal/noise ratio $\gamma = \beta$

$$\gamma = \beta$$
$$m = \alpha\beta$$

The parameterization by m and γ implies that the degenerate one-point distribution of μ in a value m_0 may be obtained as limiting distribution for Gamma distributions with mean m_0 and signal/noise ratios $\gamma \to 0$. Moreover, under that limiting process the corresponding marginal distribution of Y (negative binomial) will converge towards a Poisson distribution with mean m_0.

Conjugate prior distributions

Definition (Standard conjugate distribution for an exponential dispersion family)

Consider an exponential dispersion family $\mathrm{ED}(\mu, V(\mu)/\lambda)$ for $\theta \in \Omega$. Let $\mathcal{M} = \tau(\Omega)$ denote the mean value space for this family. Let $m \in \mathcal{M}$ and consider

$$g_\theta(\theta; m, \gamma) = \frac{1}{C(m,\gamma)} \exp\left(\frac{\theta m - \kappa(\theta)}{\gamma}\right)$$

with

$$C(m,\gamma) = \int_\Omega \exp\left(\frac{\theta m - \kappa(\theta)}{\gamma}\right) \, d\theta$$

for all (positive) values of γ for which the integral converges.
This distribution is called the *standard conjugate distribution* for θ. The concept has its roots in the context of Bayesian parametric inference to describe a family of distributions whose densities have the structure of the likelihood kernel.

Conjugate prior distributions

- When the variance function, $V(\mu)$ is at most quadratic, the parameters m and γ have a simple interpretation in terms of the mean value parameter, $\mu = \tau(\theta)$, viz.

$$m = \mathrm{E}[\mu]$$
$$\gamma = \frac{\mathrm{Var}[\mu]}{\mathrm{E}[\mathrm{Var}(\mu)]}$$

  with $\mu = \mathrm{E}[\,Y|\theta]$, and with $\mathrm{Var}(\mu)$ denoting the variance function
- The use of the symbol γ is in agreement with our introduction of γ as *signal to noise ratio* for normally distributed observations and for the Poisson-Gamma hierarchical model.

Conjugate prior distributions

- When the variance function for the exponential dispersion family is at most quadratic, the standard conjugate distribution for μ coincides with the standard conjugate distribution for θ.
- However, for the Inverse Gaussian distribution, the standard conjugate distribution for μ is improper.
- The parameterization of the natural conjugate distribution for μ by the parameters m and γ has the advantage that location and spread are described by separate parameters. Thus, letting $\gamma \to 0$, the distribution of μ will converge towards a degenerate distribution with all its mass in m.

Density for Y_i	Sufficient statistic $T(Y_1, \ldots, Y_n)$	Density for T	$\mathrm{E}[T\|\theta]$	$\mathrm{V}[T\|\theta]$
Bern(θ)	$\sum Y_i$	B(n, θ)	$n\theta$	$n\theta(1-\theta)$
B(r, θ)	$\sum Y_i$	B(rn, θ)	$rn\theta$	$rn\theta(1-\theta)$
Geo(θ)	$\sum Y_i$	NB(n, θ)	$n\frac{1-\theta}{\theta}$	$n\frac{1-\theta}{\theta}^2$
NB(r, θ)	$\sum Y_i$	NB(rn, θ)	$rn\frac{1-\theta}{\theta}$	$rn\frac{1-\theta}{\theta}^2$
P(θ)	$\sum Y_i$	P$(n\theta)$	$n\theta$	$n\theta$
P$(r\theta)$	$\sum Y_i$	P$(rn\theta)$	$rn\theta$	$rn\theta$
Ex(θ)	$\sum Y_i$	G(n, θ)	$n\theta$	$n\theta^2$
G(α, θ)	$\sum Y_i$	G$(n\alpha, \theta)$	$\alpha n\theta$	$\alpha n\theta^2$
U$(0, \theta)$	$\max Y_i$	Inv-Par(θ, n)	$\frac{n\theta}{n+1}$	$\frac{n\theta^2}{(n+1)^2(n+2)}$
N(θ, σ^2)	$\sum Y_i$	N$(n\theta, n\sigma^2)$	$n\theta$	$n\sigma^2$
N(μ, θ)	$\sum(Y_i - \mu)^2$	G$(n/2, 2\theta)$	$n\theta$	$2n\sigma^2$
N$_k(\boldsymbol{\theta}, \boldsymbol{\Sigma})$	$\sum \boldsymbol{Y}_i$	N$_k(n\boldsymbol{\theta}, n\boldsymbol{\Sigma})$	$n\boldsymbol{\theta}$	$n\boldsymbol{\Sigma}$
N$_k(\boldsymbol{\mu}, \theta\boldsymbol{\Sigma})$	$\sum(\boldsymbol{Y}_i - \boldsymbol{\mu})^T\boldsymbol{\Sigma}^-1(\boldsymbol{Y}_i - \boldsymbol{\mu})$	G$(n/2, 2\theta)$	$n\theta$	$2n\sigma^2$
N$_k(\boldsymbol{\mu}, \boldsymbol{\theta})$	$\sum(\boldsymbol{Y}_i - \boldsymbol{\mu})(\boldsymbol{Y}_i - \boldsymbol{\mu})^T$	Wis$(k, n, \boldsymbol{\theta})$	$n\boldsymbol{\theta}$	

Table: Sufficient statistic $T(Y_1, \ldots, Y_n)$ (see p. 16 in the book) given a sample of n iid random variables $Y_1, Y_2, \ldots, Y_n$. Notice that in some cases the observation is a k dimensional random vector, and here a bold notation $\boldsymbol{Y}_i$ is used.

Conditional density of T given θ	Conjugate prior for θ	Posterior density for θ after the obs. $T = t(y_1, \ldots, y_n)$	Marginal density of $T = t(Y_1, \ldots, Y_n)$
$\mathsf{B}(n, \theta)$	$\mathsf{Beta}(\alpha, \beta)$	$\mathsf{Beta}(t + \alpha, n + \beta - t)$	$\mathsf{PI}(n, \alpha, \alpha + \beta)$
$\mathsf{NB}(n, \theta)$	$\mathsf{Beta}(\alpha, \beta)$	$\mathsf{Beta}(n + \alpha, \beta + t)$	$\mathsf{NPI}(n, \beta, \alpha + \beta)$
$\mathsf{P}(n\theta)$	$\mathsf{G}(\alpha, 1/\beta)$	$\mathsf{G}(t + \alpha, 1/(\beta + n)$	$\mathsf{NB}(\alpha, \beta/(\beta + n))$
$\mathsf{G}(n, \theta)$	$\mathsf{Inv\text{-}G}(\alpha, \beta)$	$\mathsf{Inv\text{-}G}(n + \alpha, \beta + t)$	$\mathsf{Inv\text{-}Beta}(\alpha, n, \beta)$
$\mathsf{Inv\text{-}Par}(\theta, n)$	$\mathsf{Par}(\beta, \mu)$	$\mathsf{Par}(\max(t, \beta), n + \mu)$	$\mathsf{BPar}\beta, \mu, n)$
$\mathsf{N}(n\theta, n\sigma^2)$	$\mathsf{N}(\mu, \sigma_0^2)$	$\mathsf{N}(\mu_1, \sigma_1^2)$ $\mu_1 = (\mu/\sigma_0^2 + t/\sigma^2)$ $1/\sigma_1^2 = 1/\sigma_0^2 + n/\sigma^2$	$\mathsf{N}(n\mu, n\sigma^2 + n^2\sigma_0^2)$
$\mathsf{N}_k(n\boldsymbol{\theta}, n\boldsymbol{\Sigma})$	$\mathsf{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma_0})$	$\mathsf{N}_k(\boldsymbol{\mu_1}, \boldsymbol{\Sigma_1})$ $\boldsymbol{\mu_1} = \boldsymbol{\Sigma_1}(\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu} + \boldsymbol{\Sigma}^{-1}\boldsymbol{t})$ $\boldsymbol{\Sigma}_1^{-1} = \boldsymbol{\Sigma}_0^{-1} + n\boldsymbol{\Sigma}^{-1}$	$\mathsf{N}_k(n\boldsymbol{\mu}, n\boldsymbol{\Sigma} + \boldsymbol{\Sigma_0})$

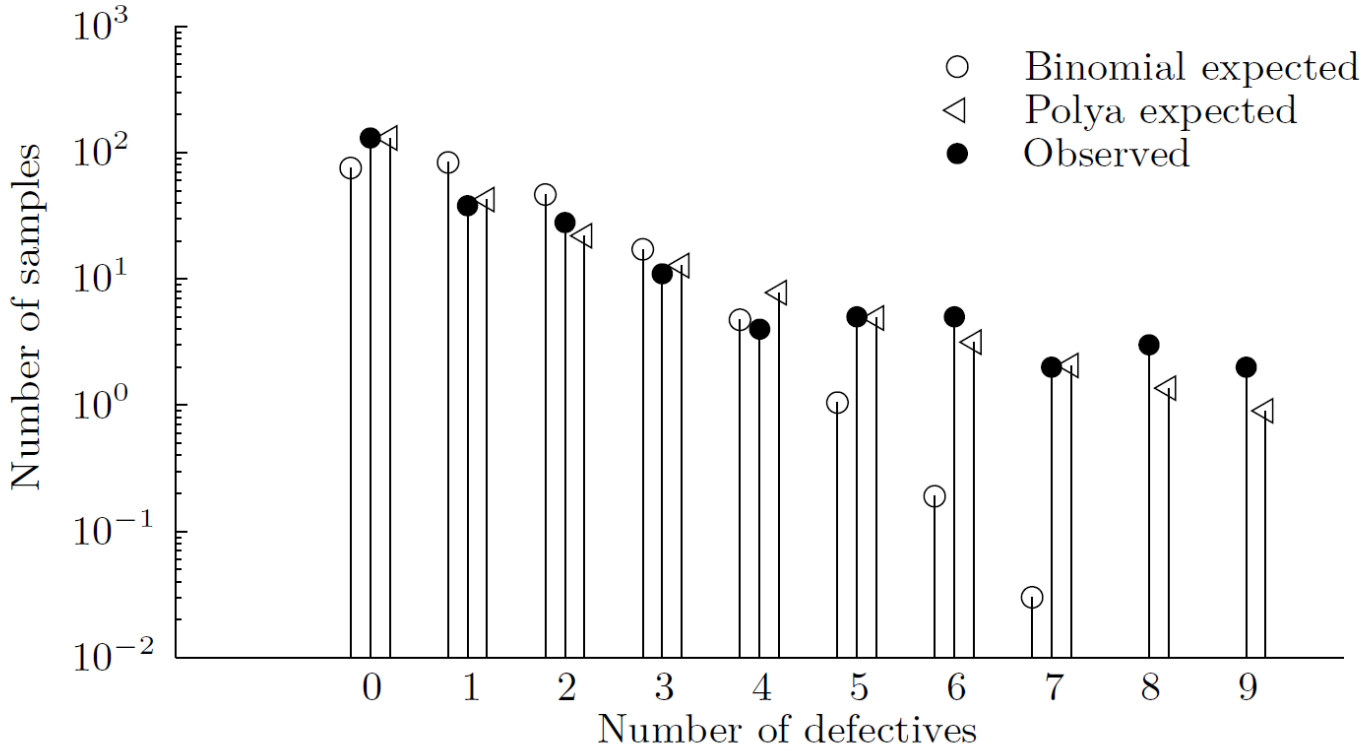
Table: Conditional densities of the statistic T given the parameter θ, conjugate prior densities for θ, posterior densities for θ after having observed the statistic $T = t(y_1, \ldots, y_n)$, and the marginal densities for $T = t(Y_1, \ldots, Y_n)$ – cf. also the discussion on page 16 and 17 in the book.(Notice that in some cases the observation is a random vector)

Hierarchical Beta-Binomial model

- Data describing the number of defective lids in samples of 770 lids from each of 229 samples.

No. defective	No. samples
0	131
1	38
2	28
3	11
4	4
5	5
6	5
7	2
8	3
9	2

- Notice that the data is summarized


Number of samples
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
0
1
2
3
4
5
6
7
8
9
Number of defectives
Binomial expected
Polya expected
Observed

Hierarchical Binomial-Beta distribution model

The natural conjugate distribution to the binomial is a Beta-distribution.

Theorem

Consider the generalized one-way random effects model for $Z_1, Z_2, \ldots, Z_k$ given by

$$\begin{aligned} Z_i|p_i &\sim B(n, p_i) \\ p_i &\sim \text{Beta}(\alpha, \beta) \end{aligned}$$

i.e. the conditional distribution of Z_i given p_i is a Binomial distribution, and the distribution of the mean value p_i is a Beta distribution. Then the marginal distribution of Z_i is a Polya distribution
with probability function

$$P[Z = z] = g_Z(z) = \binom{n}{z} \frac{\Gamma(\alpha + x)}{\Gamma(\alpha)} \frac{\Gamma(\beta + n - z)}{\Gamma(\beta)} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + n)}$$

for $z = 0, 1, 2, \ldots, n$.

Hierarchical Beta-Binomial distribution model

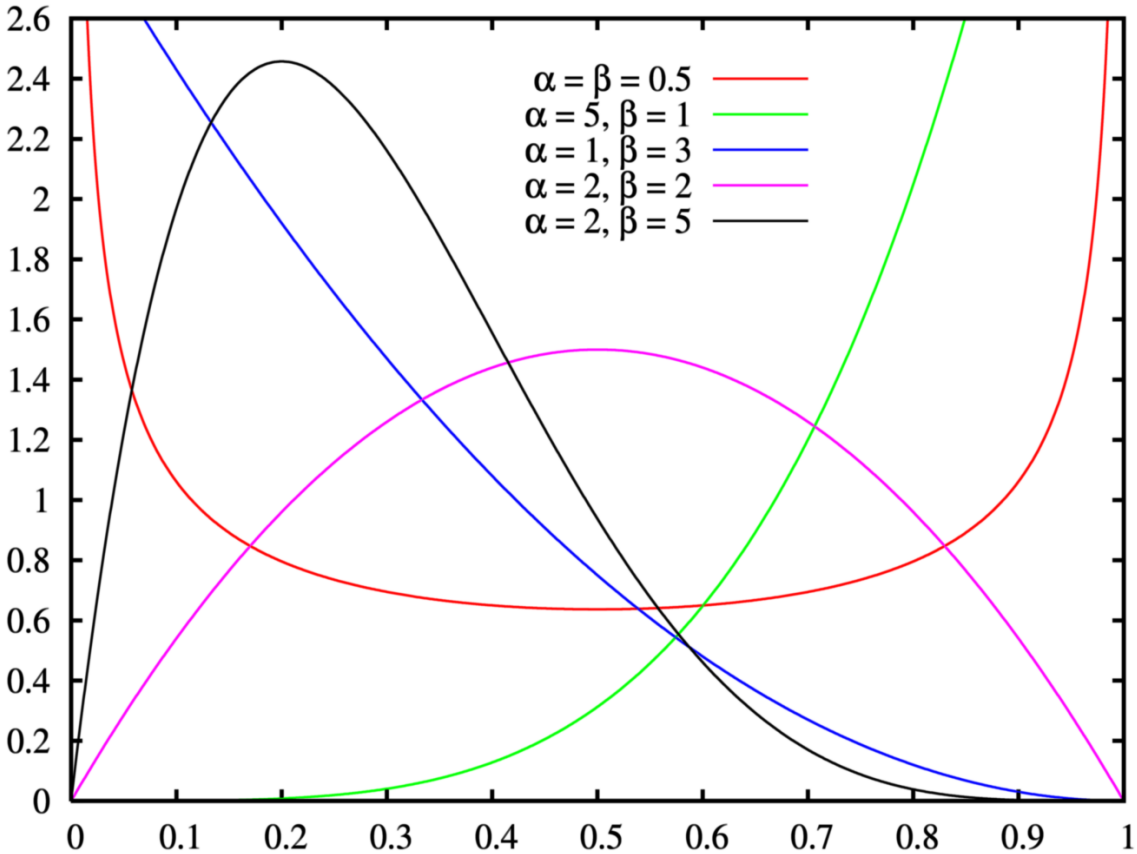
- The Polya distribution is named after the Hungarian mathematician G. Polya, who first described this distribution – although in another context.
- This distribution has:

$$\mathrm{E}[p] = \frac{\alpha}{\alpha + \beta}$$

$$\mathrm{Var}[p] = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Why use a Beta to describe variation between samples?

- It has the desired support
- It is a very flexible distribution



- Last but not least the integral can be directly calculated.

Normal distributions with random variance

As a non-trivial example (and not given in the table) of a hierarchical distribution we consider the hierarchical normal distribution model with random variance:

Theorem

Consider a generalized one-way random effects model specified by

$$Y_i|\sigma_i^2 \sim N(\mu, \sigma_i^2)$$
$$1/\sigma_i^2 \sim G(\alpha, 1/\beta)$$

where σ_i^2 *are mutually independent for* $i = 1, \ldots, k$.
The marginal distribution of Y_i *under this model is*

$$\frac{Y_i - \mu}{\sqrt{\beta/\alpha}} \sim \mathrm{t}(2\alpha)$$

where $\mathrm{t}(2\alpha)$ *is a t-distribution with* 2α *degrees of freedom, i.e. a distribution with heavier tails than the normal distribution.*

Next time

- Finish the last chapter
- Perspective: What have we learned - what more is out there