Problem Set 7

Ph.D. Course 2012:
Nodal DG-FEM for solving partial differential equations

If you have not already done so, please download all the Matlab codes from the book from

http://www.nudg.org/

and store and unpack them in a directory you can use with Matlab.

To familiarize ourselves with the discretization of higher-order operators using DG-FEM, we shall solve the nonlinear Korteweg-deVries (KdV) equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = 0, \quad x \in]\infty, \infty[\tag{1}$$

which can be used to describe localized traveling wave solution called solitons. Solitons are special wave types which can travel without change in shape due to the balance between dispersion and nonlinear effects. If two solitons travel with different speeds, they can travel past each other and without changing shape before and after they interact.

The exact solution for a right-moving soliton can be show to be given as

$$u(x,t) = f(x-ct) = \frac{c}{2}\text{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct-x_0)\right] \tag{2}$$

where x_0 is a constant of integration. The exact solution can be used for defining initial and boundary conditions as needed.

To solve the KdV equation with one spatial dimension using DG-FEM we proceed in a term-by-term fashion as follows. First, we setup a scheme for the solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in [-1, 1] \tag{3}$$

Exact solutions to the heat equation on the interval $x \in [-1, 1]$ with initial condition $u(x,t) = \sin(\pi x)$ can be found as

$$u(x,t) = \exp(-\pi^2 t)\sin(\pi x) \tag{4}$$

- Using an Energy method determine how many boundary conditions are needed and where for the heat equation to make the problem well-posed.

- Derive and implement a DG-FEM scheme for solving the heat equation on a finite domain with appropriate boundary conditions.
- Compare the computed solution with the exact solution and determine the accuracy of the scheme.

Next, we consider the dispersive equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}, \quad x \in [-1, 1] \tag{5}$$

which for an initial condition $u(x, 0) = \cos(\pi x)$ can be shown to have an exact solution given as

$$u(x, t) = \cos(\pi^3 t + \pi x) \tag{6}$$

- Using an Energy method determine how many boundary conditions are needed and where for the dispersive equation to make the problem well-posed.
- Derive and implement a DG-FEM scheme for solving the dispersive equation on a finite domain with appropriate boundary conditions.
- Compare the computed solution with the exact solution and determine the accuracy of the scheme.

Finally, we need to combine the discretization in a term-by-term fashion from the experiences with previous codes that have been setup.

- Implement a solver for the nonlinear KdV equation on a finite domain.
- Compare the computed solution with the exact solution and determine the accuracy of the scheme.
- Then, try a case where two solitons propagate at different speeds and overtake each other.

If time permits it

- What is the stable time step sizes for the KdV equation?

Enjoy!